A MULTI-CURVE LIBOR MARKET MODEL WITH UNCERTAINTIES
DESCRIBED BY RANDOM FIELDS

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<td>FRA</td>
<td>Forward rate agreement</td>
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<td>IRS</td>
<td>Interest rate swap</td>
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<td>LIBOR</td>
<td>London interbank offer rate</td>
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<td>OIS</td>
<td>Overnight indexed swap rate</td>
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<td>UKF</td>
<td>Unscented Kalman filter</td>
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<td>HVR</td>
<td>Hedging variance ratio</td>
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<td>RMS</td>
<td>Root mean square error</td>
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<td>K</td>
<td>Strike price</td>
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<tr>
<td>∆</td>
<td>Delta of the derivatives</td>
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<tr>
<td>Q</td>
<td>Risk-neutral measure, equivalent martingale measure</td>
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<td>M(t)</td>
<td>Money market account</td>
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<td>D(t,T)</td>
<td>Time-t price of stochastic discount maturating at time T</td>
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<tr>
<td>P(t,T)</td>
<td>Time-t price of a zero-coupon bond maturating at time T</td>
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<td>F(t,T,S)</td>
<td>Time-t forward interest rate with expiry T and maturity S</td>
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<tr>
<td>f(t,T)</td>
<td>Time-t instantaneous forward rate maturating at time T</td>
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<td>R(t,T)</td>
<td>Time-t spot interest rate maturating at time T</td>
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<td>L(t,T)</td>
<td>Time-t LIBOR rate maturating at time T</td>
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<td>r(t)</td>
<td>Time-t short rate</td>
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<td>Z(t)</td>
<td>A general stochastic innovation process</td>
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<tr>
<td>W^{T_j}(t)</td>
<td>Brownian motion under ( T_j )-forward measure</td>
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<td>B^Q(t)</td>
<td>Brownian motion under risk neutral measure</td>
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$Z(t, T)$ A general stochastic random field

$W^{T_j}(t, T)$ Random field under $T_j$-forward measure

$B^Q(t, T)$ Random field under risk neutral measure $Q$

$L_k(t)$ $F(t, T_{k-1}, T_k)$; Time-$t$ LIBOR forward rate on $[T_{k-1}, T_k]$

$S_{i,j}(t)$ Time-$t$ Forward swap rate for a swap with first reset date $T_i$ and payment date $T_{i+1}, ..., T_j$

$c(u, v)$ Correlation of $dW(t, u)$ and $dW(t, v)$ under appropriate measure

$\rho_{i,j}(t)$ Correlation of $dW_i(t)$ and $dW_j(t)$ under appropriate measure

$\sigma^B_k$ Black implied volatility for a caplet, in LMM

$\sigma^B_{k,RF}$ Black implied volatility for a caplet in RFLMM

$\sigma^B_{k,MR}$ Black implied volatility for a caplet in MRFLMM

$\sigma^B_{i,j}$ Black implied volatility for a swaption, in LMM

$\sigma^B_{i,j,RF}$ Black implied volatility for a swaption, in RFLMM

$\sigma^B_{i,j,MR}$ Black implied volatility for a swaption, in MRFLMM
ABSTRACT

The LIBOR (London Interbank Offered Rate) market model has been widely used as an industry standard model for interest rates modeling and interest rate derivatives pricing. In this thesis, a multi-curve LIBOR market model, with uncertainty described by random fields, is proposed and investigated. This new model is thus called a multi-curve random fields LIBOR market model (MRFLMM).

First, the LIBOR market model is reviewed and the closed-form formulas for pricing caplets and swaptions are provided. It is extended to the case when the uncertainty terms are modeled as random fields and consequently the closed-form formulas for pricing caplets and swaptions are derived. This is a new model called the random fields LIBOR market model (RFLMM).

Second, local volatility models and stochastic volatility models are combined with the RFLMM to explain the volatility skews or smiles observed in market. Closed-form volatility formulas are derived via the lognormal mixture model in local volatility case, while the approximation scheme for the stochastic volatility case is obtained by a stochastic Taylor expansion method. Moreover, the above work is further extended to a multi-curve framework, where the curves for generating future forward rates and the curve for discounting cash flows are modeled distinctly but jointly. This multi-curve methodology is recently introduced lately by some pioneers to explain the inconsistency of interest rates after the 2008 credit crunch. Both LIBOR market model and RFLMM mentioned above can be categorized as models in single-curve framework.

Third, analogous to the single-curve framework, the multi-curve random fields LIBOR market model is derived and caplets and swaptions are priced with closed-form formulas that can be reduced to exactly the Black’s formulas. This model is called a multi-curve random fields LIBOR market model (MRFLMM). Meanwhile,
local volatility and stochastic volatility models are also combined with the multi-curve LIBOR market model to explain the volatility skews and smiles in the market.

Fourth, the calibration of the above models is considered. Taking two-curve setting as an example, four different models, single-curve LIBOR market model, single-curve RFLMM, two-curve LIBOR market model and two-curve RFLMM are compared. The calibration is based on the spot market data on one trading day. The four models are calibrated to European cap volatility surface and swaption volatilities, given the specified parameterized form of correlation and instantaneous volatility. The calibration results show that the random fields models capture the volatility skews better than non-random fields models and has less pricing error. Moreover, multi-curve models perform better than single-curve models, especially during/after credit crunch.

Finally, the estimation of these four models, including pricing and hedging performance, is considered. The estimation uses time series of forward rates in market. Given a time series of term structure, the parameters of the four models are estimated using unscented Kalman filter (UKF). The results show that the random fields models have better estimation results than non-random fields models, with more accurate in-sample and out-sample pricing and better hedging performance. The multi-curve models also over-perform the single-curve models. In addition, it is shown theoretically and empirically that the random fields models have advantages that it is unnecessary to determine the number of factors in advance and not needed to re-calibrate. The multi-curve random fields LIBOR market model has the advantages of both multi-curve framework and random fields setting.
CHAPTER 1
INTRODUCTION

In this introductory chapter we briefly review the history of interest rates modeling in Sec.1.1 and introduce the framework of financial modeling and some basic financial concepts in Sec.1.2 and Sec.1.3, respectively. Then we recall the LIBOR market model (LMM) and the closed-form formulas for caplets and swaptions in Sec.1.4.

1.1 The History of Interest Rate Modeling

In the beginning, interest rate models are describing the evolution of instantaneous spot rates. There are mainly two categories, equilibrium models and no-arbitrage models. The equilibrium models define the instantaneous spot rate as a function of stationary finite-factor processes. Vasicek [68] proposed a such model, in which the evolution of instantaneous interest rate is modeled as a mean reverting process. The problem of Vasicek’s model is that it allows negative interest rates. This made Cox, Ingersoll and Ross [18] to presented a revised version, which is known as CIR model. In CIR model, the volatility term depends on the square root of the rates, to assure the positivity of interest rates. One of the main drawback of the equilibrium models is that they could not fit current term structure of interest rates, due to the fact that the drift term of the spot rate is a constant, but not a function of time. In late 1980s, a no-arbitrage model that corrected the problem was proposed by Ho and Lee [35]. In this model the drift term is assumed to be time dependent. This makes the model to fit the current term structure and to be no-arbitrage. However, this model does not have the mean reverting property. To overcome this drawback, Hull and White [38] introduced an extension to the Vasicek model. The model includes both the rates and time-dependent coefficients in the drift term. Actually, it can be characterized as the Ho-Lee model with the mean reverting property. Both Ho-Lee
and Hull-White models have the disadvantage that the short term interest rate can become negative. Black and Karasinski proposed a model which allows only positive rates, based on Hull-White model. However, it does not have as much analytic tractability as previous models. The difference of equilibrium model and no-arbitrage model can be described as in Hull [36], “In an equilibrium model, today’s term structure of interest rates is an output, while in a no-arbitrage model the term structure is an input”.

In 1992, Heath, Jarrow and Morton [31] introduced a framework (HJM framework) which described the evolution of the instantaneous forward rates, rather than the instantaneous spot rates. The benefit of using instantaneous forward rates is that the initial yield curve can be treated as an input for the model. HJM framework has been the framework of interest rate modeling ever since its establishment. However, HJM framework is hard to implement, since the state variables, the instantaneous rates, are not directly observable in the market. This makes both the model calibration and pricing of actively traded instruments difficult. Brace, Gatarek, and Musiela [12], Jamshidian [40], and Miltersen, Sandmann, and Sondermann [55] thus proposed an alternative. It is known as LIBOR (London Interbank Offered Rate) market model (LMM) or BGM model. LIBOR market model is based on the main assumption that each forward LIBOR rate follows a lognormal distribution under their own forward measure. This assumption justifies the use of Black’s formula for the pricing of interest rate sensitive options, such as caplets and swaptions. LMM has rich flexibility in choice of volatility and correlation structure and is easy to calibrate. Due to the desired features mentioned above, LMM, especially the LMM with multi-factors, is the benchmark model for interest rate modeling and derivatives pricing in the last decade.

LIBOR market model is very popular in practice, especially the LIBOR market
model with multi-factors. However, one difficult problem in interest rate modeling is how many factors should be chosen in a model. Modeling interest rate using LIBOR market model requires the determination of the number of factors in advance. A random fields interest rate model, in which the uncertainty term is modeled by random field was derived in Goldstein [26] and Kennedy [46, 47]. In this setting, it is unnecessary to determine the number of factors in advance. This is a very important feature in interest rate modeling. In this thesis, we will derive an extended LIBOR market model in which the uncertainty term is described by random field. Modeling term structure by random fields has many other advantages. By Goldstein [26], in this framework, we need only the estimation of the covariance matrix of the instantaneous forward rates to specify the model. We will show that the random fields LIBOR market model derived in this thesis also has these features.

LIBOR market model is also known for its limitation that it only generates flat implied volatility curve of caplets and swaptions, which actually display the shape of smiles or skews in real market. This will make out-of-the-money options or in-the-money options mispriced. Many works have been done to capture the smiles and skews of implied volatilities. The first popular method is to use stochastic processes that are more general than lognormal. Such models are called local volatility models. For example, constant elasticity of variance model (CEV) by Andersen and Andreasen [4] and displaced diffusion model (DD) by Joshi and Rebonato [42] generate a monotone skew but not smile of implied volatility. Another approach is to extend LIBOR market model by adopting a mean reverting square root process for variance, such as Andersen and Brotherton-Ractliffe [6], Wu and Zhang [70], which produce additional curvature to the volatility curve. Those models are called stochastic volatility models. One of the main drawbacks of above models is that the volatility dynamics of all forward rates are driven by the same stochastic process Thus it is hard to capture individual smile or skew shape of different caps and swaptions. This problem is solved by Hagan...
et al. [27] by applying SABR model to LIBOR modeling. In this thesis, we will extend the random fields LIBOR market model to local volatility models by adapting the lognormal mixture model introduced by Brigo, Mercurio and Morini [14]. It will be shown that this model creates the smile shape implied volatility and is more accurate in calibration. We will also take Wu-Zhang model and SABR model as examples of stochastic volatility model and extend the random fields LIBOR market model to the two cases.

All models derived so far become inconsistent after the credit crisis of 2007. Before the credit crisis, the market quoted interest rates are consistent. There were differences between related rates in the market, but the differences were so small that they can be regarded as negligible. However, after Aug.2007, the basis of the market rates that were consistent with each other suddenly exhibited incompatibilities. Practitioners seemed to agree on an empirical approach, which was based on the construction of as many curves as possible rate lengths(e.g. 1-month, 3-month, 6-month). Future cash flows were thus generated through the curves associated to the underlying rates and then discounted by another curve. This method is called multi-curve approach. Mercurio [54], Kijima et al. [50], Chibane et al. [17], Henrard [32], Ametrano and Bianchetti [3], Ametrano [2], and Fujii et al. [21, 22, 23] have done many pioneer works on this approach. Mercurio [54] provided a theoretical justification for the divergence of market rates. In the spirit of Kijima [50], he modeled the joint but distinct evolution of rates that refer to the same interval. Based on this approach, a multi-curve LIBOR Market Model was derived. In this thesis we will extend the random fields LIBOR market model to multi-curve framework in the spirit of Mercurio [54].

In this thesis, we will first derive an random fields LIBOR market model, in which the uncertainty term is described by random field. Second, the random fields
LIBOR market model will be extended to local volatility and stochastic volatility cases to explain the volatility smiles in the market. Third, all of the above work will be extended to multi-curve framework. Finally, the calibration and estimation of those new models will be considered. In the following we recall some basic concepts in stochastic modeling of financial markets.

1.2 Modeling Framework of Financial Market

Before we specify the interest rate model, we introduce the framework of financial market modeling. Consider a continuous financial market which consists of $n$ traded assets on the time horizon $[0, T]$. We would like to investigate the future movements of the prices of the assets. The uncertainty can be modeled by a canonical filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T \}$ and $\mathcal{F}_t$ is the $\sigma$-field generated by the innovation processes up to time $t$. The probability measure $\mathbb{P}$ is picked up from a class of equivalent probability measures on a measurable space $(\Omega, \mathcal{F}_T)$. The financial interpretation is that the investors agree on which outcomes will occur, but they have different opinions on the probabilities of these outcomes.

The price process of an asset can be modeled by a strictly positive continuous semimartingale. Mathematically a semimartingale is the most general stochastic process for which the theoretical framework has been robustly defined, for example the stochastic integral. Generally speaking, the semimartingale model is quite a natural assumption for asset price since the model can incorporate both the randomness of market information and deterministic movements.

**Definition 1.2.1. Semimartingale** [51] A regular right continuous with left limits adapted process $X(t)$ is a semimartingale if it can be represented as a sum of two
processes: a local martingale \( M(t) \), and a process of finite variation \( A(t) \),

\[
X(t) = X(0) + M(t) + A(t),
\]

with \( M(0) = A(0) = 0 \).

The \( n \) primary securities can thus be modeled as \( X(t) = \{X_1(t), \ldots, X_n(t)\} \).

In addition to the primary securities, we also need to price and hedge contingent claims. We need the theory of replicating portfolio. We first construct a self-financing trading strategy, using the primary securities, such that the final value of the portfolio equals the value of contingent claim. Then, the value of contingent claim is equal to the value of the portfolio at inception, by no-arbitrage assumption. Mathematically the above procedure can be formulated as follows. The definitions and theorems in the following literature are all quoted from Shreve [67].

**Definition 1.2.2. Arbitrage-free Self-financing Strategy**

(1) An adapted stochastic process \( S(t) = \{S_1(t), \ldots, S_n(t)\} \) is a trading strategy if \( S(t) \) is locally bounded and predictable.

(2) A trading strategy \( S(t) \) is called self-financing, if the value process \( V(t) \) satisfies the following equation

\[
V(t) = V(0) + \sum_{i=1}^{n} \int_{0}^{t} S_i(s)dX_i(s).
\]

(3) A self-financing trading strategy \( S(t) \) is called an arbitrage opportunity if the value process \( V(t) \) satisfies the following conditions

\[
V(0) = 0, \mathbb{P}(V(T) \geq 0) = 1, \mathbb{P}(V(T) > 0) > 0.
\]

The no-arbitrage framework assumes that there is no arbitrage opportunities in the market. The arbitrage-free market is defined rigorously as follows
Definition 1.2.3. Arbitrage-free Market

(1) A self-financing strategy $S(t)$ is called admissible in the market if the value process $V(t)$ is the lower bounded almost surely, i.e., there exists $K \in \mathbb{R}$ and $K < \infty$ such that

$$V(t) \geq -K, \forall t, \ a.s.$$ 

(2) A market $\mathcal{M}$ is arbitrage-free if no admissible arbitrage strategy exist in $\mathcal{M}$.

In the general framework, the prices of assets can be expressed in terms of other traded asset, so called numeraire.

Definition 1.2.4. Numeraire A numeraire is a price process $X(t)$ such that $X(t) > 0$ for all $t \in [0, T]$, almost surely.

It is desirable to choose a numeraire such that all assets are martingale under the measure defined by the numeraire. This measure is called equivalent martingale measure.

Definition 1.2.5. Equivalent Martingale Measure A equivalent martingale measure $\mathbb{Q}$ of the market $\mathcal{M}$ is a probability measure on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$ and such that all assets are martingales under $\mathbb{Q}$.

In fact we have the following theorem that establishes the equivalence of existence of equivalent martingale measure and arbitrage-free market.

Theorem 1.2.6. Absence of Arbitrary There exists an equivalent martingale measure for the market $\mathcal{M}$ if and only if the market $\mathcal{M}$ is arbitrage-free.

A market is defined to be complete if every integrable contingent claim is attainable. We have the following theorem.
Theorem 1.2.7. Market Completeness Every claim \( Y(t) \) is attainable in the market \( \mathcal{M} \) if and only if there is only one equivalent martingale measure \( \mathbb{Q} \).

All of the pricing formulas in this thesis are based on the following theorem.

Theorem 1.2.8. Equivalent Martingale Pricing Suppose that there exists an equivalent martingale measure \( \mathbb{Q} \) in market \( \mathcal{M} \). Let \( Y(t) \) be an attainable contingent claim in \( \mathcal{M} \). Then the price process \( V(t) \) of the claim is given by

\[
V(t) = \mathbb{E}^\mathbb{Q}[Y(T)|\mathcal{F}(t)].
\]

If \( \mathbb{Q} \) is an equivalent martingale measure that is obtained from \( \mathcal{M} \) under a change of numeraire \( P(t) \), then the price process \( V(t) \) is given by

\[
V(t) = P(t)\mathbb{E}^\mathbb{Q}\left[\frac{Y(T)}{P(T)}|\mathcal{F}(t)\right].
\]

1.3 Basic Financial Concepts

In this section we provide some basic concepts that are used throughout this thesis. The definitions in the following literature are all quoted from Brigo and Mercurio [13].

Definition 1.3.1. Money Market Account (Bank Account). The value of money market account at time \( t \), \( M(t) \), evolves according to the differential equation,

\[
dM(t) = r(t)M(t)dt, M(0) = 1,
\]

where \( r(t) \) is a positive function of time. As a consequence,

\[
M(t) = e^{\int_0^t r(s)ds}.
\]

In this definition, \( r(t) \) is the instantaneous rate at which the money market account accrues. This rate is referred as the instantaneous spot rate or short
rate. The money market account $M(t)$ relates the amount of currencies at different time instants. In fact $M(t)/M(T)$ represents the amount at time $t$ that is equivalent to one unit of currency payable at time $T$. This amount can be denoted as stochastic discount factor $D(t,T)$, which is equal to $e^{-\int_t^T r(s)ds}$.

**Definition 1.3.2. Zero Coupon Bond.** A $T$-maturity zero coupon bond is a contract that guarantees its holder the payment of one unit of currency at time $T$, without any intermediate payments. The value at time $t < T$ of the contract is denoted by $P(t,T)$.

Zero coupon bond $P(t,T)$ can be actually viewed as the expectation of the random variable $D(t,T)$ under a particular probability measure. In particular if the rate $r(t)$ are deterministic, we have $P(t,T) = D(t,T)$. In moving from zero coupon bond to interest rates, we need to specify two features of the rates, the compounding type and the day count convention. For simplicity we just use $\delta(t,T)$ to denote the amount of time (in years) between time $T$ and time $t$, with preferred day count convention.

**Definition 1.3.3. Spot Interest Rate.** The spot interest rate $R(t,T)$ is the constant rate at which an investment of $P(t,T)$ unit of currency at time $t$ accrues to yield a unit amount of currency at maturity at $T$. For different compounding type, we have

- **continuously compounded:** $P(t,T) = e^{-R(t,T)\delta(t,T)}$.

- **simply compounded:** $P(t,T) = \frac{1}{1+R(t,T)\delta(t,T)}$.

- **$k$-times per year compounded:** $P(t,T) = \frac{1}{(1+R(t,T)/k)^{\delta(t,T)}}$.

In fact, the short rate is a limit of spot rate, $r(t) = \lim_{T \to t^+} R(t,T)$. The market LIBOR rates $L(t,T)$ are simply-compounded spot rates linked to Actual/360
day count convention. To define the instantaneous forward rate which we will use as our fundamental underlying process, we introduce forward rate.

**Definition 1.3.4. Forward Interest Rate.** The forward interest rate $F(t, T, S)$ at time $t$ for expiry $T > t$ and maturity $S > T$ is defined by

- **continuously compounded:** $F(t, T, S) = -\frac{\ln P(t; T; S)}{\delta(t, S)}$.

- **simply compounded:** $F(t, T, S) = \frac{1}{\delta(t, S)} \left( \frac{P(t, T)}{P(t, S)} - 1 \right)$.

The instantaneous forward interest rate $f(t, T)$ can be defined as the limit of forward rate

$$f(t, T) = \lim_{S \to T^+} F(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T}. \tag{1.1}$$

Thus we have

$$P(t, T) = e^{-\int_t^T f(t, u) \, du}. \tag{1.2}$$

And the discounted bond price is given as

$$P(t, T)e^{-\int_0^t r(s) \, ds}. \tag{1.3}$$

We now introduce several main derivative products of the interest rate market, Forward Rate Agreement (FRA), Interest Rate Swap (IRS), Caps, Swaptions, as well as the relative rates (FRA rates, Swap rates).

**Definition 1.3.5. Forward Rate Agreement.** A Forward Rate Agreement (FRA) is a contract giving the holder an interest rate payment for the period between expiry $T$ and maturity $S$. At time $S$, a fixed payment $K$ based on a fixed rate $L(T, S)$ is exchanged against a floating payment based on spot rate $L(T, S)$ resetting in $T$ and with maturity $S$. 
The value of FRA at time $t$, with contact nominal value $A$, is given as

$$\text{FRA}(t, T, S, A, K) = A[P(t, S)\delta(T, S)K - P(t, T) + P(t, S)].$$  \hspace{1cm} (1.4)$$

The value of fixed rate $K$ that renders the FRA a fair contract at time $t$ is thus

$$\frac{1}{\delta(T, S)}\left(\frac{P(t, T) - P(t, S)}{P(t, S)}\right),$$

which is exactly forward rate $F(t, T, S)$. Thus we also call $F(t, T, S)$ as FRA rate. In fact the forward rate $F(t, T, S)$ can also be defined as the expectation of $L(T, S)$ at time $t$ under a suitable probability measure.

**Definition 1.3.6. Interest Rate Swap.** A (forward-start) Interest Rate Swap (IRS) is a contract that exchanges payments between two indexed legs, starting from a future time. At every $T_k \in [T_{i+1}, ..., T_j]$, the fixed-legs pay $A\delta_kK$, corresponding a fixed rate $K$, a nominal $A$ and year fraction $\delta_k = T_k - T_{k-1}$, whereas the floating-leg pays $A\delta_kL(T_{k-1}, T_k)$, corresponding to $L(T_{k-1}, T_k)$ resetting at $T_{k-1}$ and maturity $T_k$.

When the fixed-leg is received and the floating-leg is paid, the IRS is called receiver IRS, whereas in the other case we have a Payer IRS. Notice that the floating-leg rate resets at dates $T_i, ..., T_{j-1}$ and pays at days $T_{i+1}, ..., T_j$, thus we say that IRS has time set $\mathcal{T}$, with $\mathcal{T} = \{T_i, ..., T_j\}$. It is clear that IRS is a generalization of FRA, thus the value of a receiver IRS at time $t$ is given as

$$\text{IRS}(t, \mathcal{T}, A, K) = \sum_{k=i+1}^{j} \text{FRA}(t, T_{k-1}, T_k, A, K)$$

$$= A \left[ \sum_{k=i+1}^{j} \delta_k P(t, T_k)F(t, T_{k-1}, T_k) - \sum_{k=i+1}^{j} \delta_k K P(T, t_k) \right]$$

$$= A[P(t, T_j) - P(t, T_i) + \sum_{k=i+1}^{j} \delta_k K P(T, t_k)].$$  \hspace{1cm} (1.5)$$

The value of fixed rate $K$ that renders the FRA a fair contract at time $t$ is

$$\frac{\sum_{k=i+1}^{j} \delta_k P(t, T_k)F(t, T_{k-1}, T_k)}{\sum_{k=i+1}^{j} \delta_k P(T, t_k)} = \frac{P(t, T_i) - P(t, T_j)}{\sum_{k=i+1}^{j} \delta_k P(T, t_k)},$$

which is termed as swap rate $S_{i,j}(t)$. 

We conclude this section by introducing two main interest rate derivatives, caps and swaptions. We use the market price of these two derivatives to calibrate and estimate the models in this thesis.

**Definition 1.3.7. Interest Rate Caps.** *Interest rate caps are agreements to borrow money at some maximum interest rate for a given period. It can be viewed as a payer IRS where each exchange payment is executed only if it has positive value. It consists of a series of called option on LIBOR forward rate.*

Suppose the maximum interest rate is $K$ and the period is $T_i, ..., T_j$, the cap discounted payoff at time $t$ is given as

$$
\sum_{k=i+1}^{j} AD(t, T_k)\delta_k(L(T_{k-1}, T_k) - K)^+, \quad (1.7)
$$

where $(X)^+$ means the maximum of $X$ and 0.

**Definition 1.3.8. Interest rate swaptions** *Interest rate swaptions give its owner the right, but not the obligation, to enter into a certain IRS at the pre-negotiated strike rate. It can be seen as an option on the swap rate.*

Suppose the strike rate is $K$ and expiration time is $T_i$, the discounted payoff of swaption at time $t$ is

$$
AD(t, T_i)(\sum_{k=i+1}^{j} P(T_i, T_k)\delta_k(F(T_i, T_{k-1}) - T_k)^+) +, \quad (1.8)
$$

or

$$
AD(t, T_i)(S_{i,j}(T_i) - K)^+ \sum_{k=i+1}^{j} P(T_i, T_k)\delta_k. \quad (1.9)
$$

From the payoff formulas we can see that the fundamental difference of these two derivatives is that the payoff of cap can be decomposed into more elementary products. In this thesis, we derive the pricing formulas of these two derivatives with different models and use the formulas to calibrate or estimate the models.
We assume that \( f(t, T) \) is driven by the following diffusion process:

\[
df(t, T) = \mu(t, T) dt + \sigma(t, T) \cdot dZ(t); \quad f(0, T) = f_0(T).
\] (1.10)

The process \( \mu(t, \omega, \cdot) : [0, T] \times \Omega \to \mathbb{R} \) and \( \sigma(t, \omega, \cdot) : [0, T] \times \Omega \to \mathbb{R}^d \) may be stochastic and are assumed to be locally bounded, locally Lipschitz continuous and predictable, to make sure the existence and uniqueness of the solution. \( Z(t) \) is a \( d \)-dimension stochastic innovation process driving the diffusion process. In this thesis we consider only Brownian motion \( W(t) \) and its extension random field \( W(t, T) \). The definition of random field \( W(t, T) \) will be given in Chapter 2.

1.4 LIBOR Market Model (LMM)

In this section we review the LIBOR market model (LMM), which was constructed in the HJM framework by Brace, Gatarek, and Musiela [12]. We first provide an overview of HJM framework built by Heath, Jarrow and Morton [31], which is the basis of further derivation in this thesis.

**Heath-Jarrow-Morton Framework** Heath, Jarrow and Morton [31] built a theoretical framework expressing the absence of arbitrage using \( f(t, T) \) as fundamental quantities to be modeled. For the zero coupon bond price \( P(t, T) \) defined in Eq.(1.2), if we require that the discounted bond price Eq.(1.3) is a martingale under risk neutral measure \( Q \), where \( r(t) \) is the instantaneous spot rate, the no-arbitrage drift term \( \mu(t, T) \) must be given by \( \sigma(t, T) \cdot \int_t^T \sigma(t, u) du \). The dynamics of instantaneous forward rate under risk neutral measure \( Q \) is thus given by

\[
df(t, T) = \sigma(t, T) \cdot \int_t^T \sigma(t, u) du dt + \sigma(t, T) \cdot d\overline{W}(t),
\] (1.11)

where \( \overline{W}(t) \) is a \( d \)-dimensional Brownian motion under risk neutral measure \( Q \) and \( \sigma(t, u) \) is a \( \mathbb{R}^d \)-valued, adapted process. The existence and uniqueness of solution \( f(t, T) \) in Eq.(1.11) is assured by Theorem A.9, given the coefficients satisfying the required conditions, locally bounded, locally Lipschitz continuous and predictable.
Given the dynamics of $f(t, T)$ in Eq.(1.11), by Itô’s formula we can derive that the dynamics of zero coupon bond is

$$dP(t, T) = P(t, T)[r(t)dt - \sigma^*(t, T) \cdot d\tilde{W}(t)],$$

where $\sigma^*(t, T) = \int_t^T \sigma(t, u)du$.

Let us consider the time structures $\{T_0, T_1, \ldots, T_N\}$ with time intervals $\delta_k = T_k - T_{k-1}, k = 1, \ldots, N$. For $t < T_{k-1} < T_k$, the LIBOR forward rate $L_k(t)$ is defined as

$$L_k(t) = \frac{1}{\delta_k} \frac{P(t, T_{k-1})}{P(t, T_k)} - 1].$$ (1.12)

The dynamics of $L_k(t)$ is determined by those of zero coupon bonds, thus by Itô’s formula we can derive that

$$dL_k(t) = L_k(t)\lambda_k(t) \cdot [\sigma^*(t, T_k) dt + d\tilde{W}(t)],$$ (1.13)

where $\lambda_k(t) = 1 + \frac{\delta_k L_k(t)}{\delta_k L_k(t)} \int_{T_{k-1}}^{T_k} \sigma(t, u)du = \frac{1}{\delta_k L_k(t)} [\sigma^*(t, T_k) - \sigma^*(t, T_{k-1})]$. By equivalent martingale measure theorem, since LIBOR forward rate $L_k(t)$ defined in Eq.(1.12) use zero coupon bond $P(t, T_k)$ as the unit of measurement, $L_k(t)$ should be a martingale under the measure that uses $P(t, T_k)$ as a numeraire. By Eq.(1.13), a so called $T_k$-forward measure is defined such that under which

$$\int_0^t \sigma^*(s, T_k)ds + \tilde{W}(t)$$ (1.14)

is a $d$-dimensional Brownian motion. We denote it as $W_{T_k}(t)$. Thus we conclude that $L_k(t)$ is a martingale and follows lognormal distribution under $T_k$-forward measure:

$$dL_k(t) = L_k(t)\lambda_k(t) \cdot dW_{T_k}(t).$$ (1.15)

Using the relation

$$dW_{T_k}(t) - \sigma^*(t, T_k)dt = d\tilde{W}(t) = dW_{T_k-1}(t) - \sigma^*(t, T_{k-1})dt$$
\[ \lambda_k(t) = \frac{1 + \delta_k L_k(t)}{\delta_k L_k(t)} [\sigma^*(t, T_k) - \sigma^*(t, T_{k-1})], \]

we can derive the dynamics of \( L_k(t) \) under \( T_j \)-forward measure, in three cases \( j < k, j = k, j > k \), respectively,

\[
\begin{align*}
\text{for } j < k, & \quad dL_k(t) = L_k(t) \lambda_k(t) \cdot [dW^T_j(t) + \sum_{i=j+1}^{k} \frac{\delta_i L_i(t) \lambda_i(t)}{\delta_i L_i(t) + 1} dt], \\
\text{for } j = k, & \quad dL_k(t) = L_k(t) \lambda_k(t) \cdot dW^T_j(t), \\
\text{for } j > k, & \quad dL_k(t) = L_k(t) \lambda_k(t) \cdot [dW^T_j(t) - \sum_{i=k+1}^{j} \frac{\delta_i L_i(t) \lambda_i(t)}{\delta_i L_i(t) + 1} dt].
\end{align*}
\]

From Eq. (1.15), we know that \( L_k(t) \) is lognormal distributed under \( T_k \)-forward measure.

Actually we can rewrite Eq. (1.16) in a different form. Assume that \( L_k(t) \) have dynamics

\[ dL_k(t) = \xi_k(t) L_k(t) dW^T_k(t), \]

for \( k = 1, 2, ..., N \), where \( W^T_k(t) \) is the \( k \)-component of \( N \)-dimensional Brownian motion under \( T_k \)-forward measure and \( corr(dW^T_i(t), dW^T_j(t)) = \rho_{i,j}(t) \). In this case \( \xi_k(t) \) is a scalar function. From Sec.2.3 we know that the dynamics (1.17) is a discrete case of random fields model, which models instantaneous forward rate \( f(t, T) \) as in Eq.(2.6). To derive the drift term of \( dL_k(t) \) under \( T_j \)-forward measure, we use the change of measure techniques as in Brigo and Mercurio [13]. For \( j < k \), the drift term is

\[ -dL_k(t)d\ln \frac{P(t, T_k)}{P(t, T_j)} = -dL_k(t) \ln \left( \frac{1}{\prod_{i=j+1}^{k} (1 + \delta_i L_i(t))} \right) \]

\[ = \sum_{i=j+1}^{k} \frac{\delta_i}{1 + \delta_i L_i(t)} dL_k(t) dL_i(t) dt \]

\[ = L_k(t) \xi_k(t, u) \sum_{i=j+1}^{k} \frac{\delta_i L_i(t) \xi_i(t, u) \rho_{i,k}(t)}{1 + \delta_i L_i(t)}. \]
The derivation in case \( k < j \) is perfectly analogous. Thus the dynamics of \( L_k(t) \) under \( T_j \)-forward measure, in three cases \( j < k, j = k, j > k \), are described respectively by the following equations

\[
\begin{align*}
    dL_k(t) &= L_k(t)\xi_k(t)[dW_k^{T_j}(t) + \sum_{i=j+1}^{k} \frac{\delta_i \rho_{i,k}(t)L_i(t)\xi_i(t)}{\delta_i L_i(t) + 1} dt], & j < k; \\
    dL_k(t) &= L_k(t)\xi_k(t)dW_k^{T_j}(t), & j = k; \\
    dL_k(t) &= L_k(t)\xi_k(t)[dW_k^{T_j}(t) - \sum_{i=k+1}^{j} \frac{\delta_i \rho_{i,k}(t)L_i(t)\xi_i(t)}{\delta_i L_i(t) + 1} dt], & j > k.
\end{align*}
\] (1.18)

Kerkhof and Pelsser [48] showed that the discrete random fields model Eq.(1.18) and market model Eq.(1.16) are equivalent, given the number of factor in market model is the same as the state dimension of discrete random fields model, i.e, \( k = 1, 2, 3, \ldots, d \).

Indeed, given a decomposition of the correlation matrix \( \rho(t) = Q(t)Q'(t) \), where \( Q(t) \in \mathbb{R}^{d \times d} \), we can express the first equation in Eq.(1.18) as

\[
dL_k(t) = L_k(t)\xi_k(t)[q_k'(t) \cdot dW^{T_j}(t) + \sum_{i=j+1}^{k} \frac{\delta_i q_k'(t) \cdot q_i'(t)L_i(t)\xi_i(t)}{\delta_i L_i(t) + 1} dt],
\] (1.19)

where \( q_k(t) \) is the \( k \)-th row of \( Q(t) \). The relationship between two formulation is clear:

\[
\xi_k(t)q_k'(t) = \lambda_k(t).
\] (1.20)

In this thesis we use Eq.(1.18) instead of Eq.(1.16) as our LIBOR market model. In this case, the number of factors in LIBOR market model depends on the rank of correlation matrix \( \rho \).

1.5 Option Pricing in LIBOR Market Model

In this section we review the derivation of closed-form formulas for pricing European caplets and swaptions in LIBOR market model.

1.5.1 LIBOR market formula for caplets. Interest rate caps are agreements to borrow money at some maximum interest rate \( K \) for a given period, \( T_1, \ldots, T_N \). It
consists of a series of derivatives called caplets. A caplet is a call option on a LIBOR forward rate. A caplet gives its owner the right, but not the obligation, to borrow money over the forward accrual period at the pre-negotiated strike rate. The caplet payoff is paid out at the end of the forward accrual period. The payoff of caplets at time $T_k$ is

$$\delta_k (L_k(T_{k-1}) - K)^+,$$

where $(X)^+$ means the maximum of $X$ and 0. And the time $t$ price of a caplet is

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E_{T_k}[(L_k(T_{k-1}) - K)^+ | \mathcal{F}_t].$$

In the LIBOR market model, the LIBOR forward rates are assumed to be log-normally distributed under the associated forward measure and have volatility $\xi_k(t)$, as shown in Eq.(1.15). By using Black’s formula derived in Black [11], the time $t$ price of caplets can be derived as

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \text{Black}(K, L_k(t), \sigma_k^{\text{Black}} \sqrt{T_{k-1} - t}),$$

where

$$\text{Black}(K, L_k(t), X) = L_k(t) N(d_1) - K N(d_2),$$

and

$$d_1 = \frac{\log(L_k(t)/K) + X^2/2}{X},$$
$$d_2 = \frac{\log(L_k(t)/K) - X^2/2}{X} = d_1 - X.$$

Prices of caplets are actually quoted in terms of

$$\sigma_k^{\text{Black}}(t) := \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \| \xi_k(s) \|^2 \, ds},$$

which are called Black implied volatilities. The Black implied volatility of a caplet is the volatility that returns the market quoted price using Black formula, with preferred choice of discount factor $P(t, T_k)$. 
1.5.2 LIBOR market formula for swaptions. A swap is an agreement between two parties to swap fixed for floating interest rate payments on same notional amount. The floating interest rate may for instance be LIBOR rates which are set at the beginning of the accrual period and the fixed interest rate are determined by the agreement. The payment is made at the end of each accrual period. The fixed rate at which the swap has zero value is called swap rate.

It can be shown that swap rate \( S_{i;j}(t) \) with expiration time \( T_i \) and payment times \( T_{i+1}, \ldots, T_j \) is

\[
S_{i;j}(t) = \frac{\sum_{k=i+1}^{j} \delta_k P(t, T_k) L_k(t)}{\sum_{k=i+1}^{j} \delta_k P(t, T_k)} = \frac{P(t, T_i) - P(t, T_j)}{\sum_{k=i+1}^{j} \delta_k P(t, T_k)},
\]

for \( 0 \leq t \leq T_i, \ i < j \leq N \).

A swaption could be seen as an option on the swap rate. A swaption gives its owner the right, but not the obligation, to enter into a certain swap at the pre-negotiated strike rate. Suppose that the strike rate is \( K \) and expiration time is \( T_i \), then the payoff of swaption at time \( T_k \) is

\[
\delta_k (S_{i;j}(T_k - 1) - K)^+.
\]

In the swap market model, the swap rates are assumed to have the dynamics

\[
dS_{i;j}(t) = S_{i;j}(t) \eta_{i;j}(t) dW^{i;j}(t).
\]

And the time \( t \) price of swaption can be derived by Black’s formula as

\[
\text{Swpt}(t, K, T_i, T_j) = A \text{ Black}(K, S_{i;j}(t), \sigma_{i;j}^{Black} \sqrt{T_{k-1} - t}),
\]

where \( \sigma_{i;j}^{Black} := \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \| \eta_{i;j}(s) \|^2 ds, \ A = \sum_{k=i+1}^{j} \delta_k P(t, T_k).} \)

The simultaneous assumption of lognormal distributed forward rates and lognormal distributed swap rates is not consistent. Conclusively swaption can not be
priced using Black’s formula within LIBOR market model. Now we will rewrite the implied volatility of swaption in terms of LIBOR rates $L_k(t)$. Given

$$P(t, T_k) = \frac{1}{1 + \delta_j L_j(t)}$$

for $k \geq i + 1$, by dividing through $P(t, T_j)$, the swap rate defined in Eq.(1.25) can be written as

$$S_{i,j}(t) = \frac{\prod_{l=i+1}^{j} (1 + \delta_l L_l(t)) - 1}{\sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_l(t))},$$

or equivalently

$$\ln S_{i,j}(t) = \ln\left\{ \prod_{l=i+1}^{j} (1 + \delta_l L_l(t)) - 1 \right\} - \ln\left\{ \sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_l(t)) \right\},$$

for $j - 1 \geq k \geq i + 1$, where $\prod_{m=1}^{n} = 1$ if $m > n$. According to Hull et al.[39] and from Itô’s formula, the uncertainty part of swap rates in LMM model can be derived as:

$$\sum_{k=i+1}^{j} \frac{1}{S_{i,j}(t)} \frac{\partial S_{i,j}(t)}{\partial L_k(t)} dW^T_k(t) = \sum_{k=i+1}^{j} \frac{\delta_k L_k(t) \gamma_{i,j}^{k}(t)}{1 + \delta_k L_k(t)} \xi_k(t) dW^T_k(t),$$

where

$$\gamma_{i,j}^{k}(t) = \frac{\prod_{l=i+1}^{j} (1 + \delta_l L_l(t))}{\prod_{l=i+1}^{j} (1 + \delta_l L_l(t)) - 1} - \frac{\sum_{m=i+1}^{k-1} \delta_m \prod_{l=m+1}^{j} (1 + \delta_l L_l(t))}{\sum_{m=i+1}^{k} \delta_m \prod_{l=m+1}^{j} (1 + \delta_l L_l(t))}.$$

Swaptions are usually quoted in Black implied volatilities, which are defined approximately as

$$\sigma_{i,j}^{Black}(t) = \frac{1}{T_t - t} \int_t^{T_t} \left\| \sum_{k=i+1}^{j} \frac{\delta_k L_k(s) \gamma_{i,j}^{k}(s)}{1 + \delta_k L_k(s)} \xi_k(s) \right\|^2 ds$$

$$= \sqrt{\frac{1}{T_t - t} \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\delta_k L_k(t) \gamma_{i,j}^{k}(t) \delta_l L_l(t) \gamma_{i,j}^{l}(t)}{1 + \delta_k L_k(t)} \frac{1}{1 + \delta_l L_l(t)}}$$

$$\times \int_t^{T_t} \rho_{kl}(s) ||\xi_k(s)|| ||\xi_l(s)|| ds.$$  

(1.28)

Prices of swaption are actually quoted in terms of $\sigma_{i,j}^{Black}$, which are called Black implied volatilities. Approximatively, The Black implied volatility of a swaption is the volatility that returns the market quoted price using Black formula, with preferred choice of discount factors $P(t, T_i),...,P(t, T_j)$. 


Remark 1.5.1. Standard Freezing Approximation Techniques.

The last equation is obtained by approximatively evaluating the LIBOR rate \( L_k(s) \), \( t \leq s \leq T_i \) at initial time \( t \). This approximation technique is shown to have very high accuracy as shown by Hull and White [38]. They compared the prices of swaptions calculated by the approximation formula above with the price calculated from a Monte Carlo simulation and found the two to be very closely related. We will use this approximation frequently in this thesis.

The rest of this thesis is organized as follows. Chapter 2 introduces a new LIBOR market model with random fields setting and caplets and swaptions are priced in this model. Chapter 3 investigates the volatility smile modeling and establishes volatility smile models with random fields setting. Chapter 4 extends all the previous works to multi-curve setting, where the curve for generating future forward rates and the curve for discounting cash flows are different. Chapter 5 is about the calibration and relevant numerical results. Chapter 6 discusses the estimation and corresponding numerical results. Finally, Chapter 7 provides the conclusions of this thesis.
In this chapter we derive an extended LIBOR market model with uncertainties described as random fields, which is termed as random fields LIBOR market model (RFLMM). First, we introduce random fields as a description of uncertainty in Sec.2.1. Second, we review the advantages of modeling interest rates as random fields in Sec.2.2. Third, we derive the random fields LIBOR market model in Sec.2.3, as well as the closed-form formulas for pricing European caplets and swaptions.

2.1 Random Fields

A random field is a stochastic process that is indexed by a spatial variable, as well as a time variable. For example, if we would like to measure the temperature at position \( u \) and time \( t \) with \( u \in \mathbb{R}^n \) and \( t \in \mathbb{R}^+ \), the measure can be modeled as a random variable \( W(t, u) \). The collection of

\[
\{W(t, u) : (t, u) \in \mathbb{R}^+ \otimes \mathbb{R}^n\}
\]

is a random field.

Following the construction procedure of random field in Bester [8], we begin the definition of random field from Brownian field. Standard one-dimensional Brownian motion \( W(t) \) can be constructed as

\[
W(t) = W(0) + \int_0^t \epsilon(s)ds.
\]  

(2.1)

Here \( \epsilon(t) \), with \( t \in [0, \infty] \), is a scalar white noise process, i.e., \( \mathbb{E}[\epsilon(t)] = 0 \) and \( \text{cov}[\epsilon(t), \epsilon(s)] = \delta(t - s) \), with Delta function \( \delta(t - s) \). We can similarly define a two-dimensional white noise process \( \epsilon(t, u) \), with \( (t, u) \in [0, \infty) \times \mathbb{R} \), by \( \mathbb{E}[\epsilon(t, u)] = 0 \) and \( \text{cov}[\epsilon(t, u), \epsilon(s, v)] = \delta(t - s)\delta(u - v) \).
The Browian field can thus be defined by

\[ W(t, u) = W(0, 0) + \frac{1}{\sqrt{u}} \int_0^u \int_0^t \epsilon(s, x) ds dx. \] (2.2)

The construction of Browian field \( W(t, u) \) can be generalized to random fields \( Z(t, u) \), given a specification of the correlation structure of the field increments \( c(u, v) \):

\[ c(u, v) = \text{corr}[dZ(t, u), dZ(t, v)], \] (2.3)

where the differential notation \( d \) is used to denote the increments in \( t \)-direction. The random field \( W(t, u) \) can be constructed as follows. Assume that for a correlation function \( c(u, v) \), there exists a symmetric function \( g(u, v) \) such that

\[ c(u, v) = \int_0^\infty g(u, z) g(v, z) dz, \]

with \( \int_0^\infty |g(u, z)|^2 dz = 1 \). The random field \( Z(t, u) \) can be defined as

\[ Z(t, u) = Z(0, 0) + \int_0^\infty \int_0^t g(u, z) \epsilon(s, z) ds dz, \] (2.4)

or equivalently

\[ dZ(t, u) = [\int_0^\infty g(u, z) \epsilon(t, z) dz] dt \] (2.5)

From Eq.(2.5) we can see that the increments of random field are weighted average of white noise at time \( t \). Here \( g(u, z) \) is the weight of \( \epsilon(t, z) \) at location \( z \) in determining the change of the field at location \( u \). In this thesis, we use \( W(t, T) \) to denote random field.

A formal theoretic definition of random field and other treatments can be found in Adler [1], Gikhman-Skorokhod [25] and Khoshnevisan [49]. In this definition, most of the concepts and analysis on Brownian motion can be similarly applied to random field, such as the Itô integral, quadratic variation, stochastic differentials, and stochastic differential equations, among others. See Appendix A. Further discussions of stochastic dynamics can be found in Duan [19].
2.2 Interest Rate Modeling in Random Fields Setting

Modeling interest rates as random fields was introduced by Kennedy [46, 47] and Goldstein [26]. Limiting the scope to Gaussian random fields, Kennedy [46] obtained the form of the drift terms of the instantaneous forward rates processes necessarily to preclude arbitrage under risk neutral measure and Goldstein [26] extended the work to the case of non-Gaussian random fields. We first provide an overview of the Kennedy-Goldstein Framework.

Kennedy-Goldstein Framework. According to Goldstein [26], for zero coupon bond price $P(t, T)$ defined in Eq.(1.2), if we require that the discounted bond price Eq.(1.3) to be a martingale under risk neutral measure $Q$, the no-arbitrage drift term must be given by $\sigma(t, T) \int_t^T \sigma(t, u)c(t, T, u)du$. Thus the dynamics of the instantaneous forward rates $f(t, T)$ under risk neutral measure is given as

$$df(t, T) = \sigma(t, T) \int_t^T \sigma(s, u)c(s, T, u)duds + \sigma(t, T)d\tilde{W}(t, T),$$

(2.6)

with the correlation structure

$$\text{corr}[dW(t, T_1), dW(t, T_2)] = c(t, T_1, T_2),$$

(2.7)

where $\tilde{W}(t, T)$ is a random field under risk neutral measure $Q$ and $\lim_{\Delta T \to 0} c(t, T, T + \Delta T) = 1$. The existence and uniqueness of solution $f(t, T)$ in Eq.(2.6) is assured by Theorem A.9, given the coefficients satisfying the required conditions, locally bounded, locally Lipschitz continuous and predictable.

Heath, Jarrow and Morton [31] proved the existence of risk neutral measure directly and built up a framework to pricing all contingent claims. Rather than identify the risk neutral measure, Goldstein [26] first assumes its existence and derives the dynamics of the instantaneous forward rates under this measure, then shows the existence of the risk neutral measure within a general equilibrium framework. By
Goldstein [26] and Pang [60], modeling term structure as random fields has many advantages. First, it is unnecessary to determine the number of factors prior to the calibration or estimation of the random fields model. This feature is very important and crucial, since one of the most difficult problems in interest rate model is the determination of factors before model building. In fact, random fields models are infinite-factor models, since the instantaneous forwards in random fields models form a continuum. Second, the re-calibration existing among no-arbitrage models is unnecessary in random fields models. Re-calibration violates the no-arbitrage feature, which are the fundamental assumption of the models. We now discuss the above advantages in details.

First, determining the number of factors prior to the calibration or estimation is not required for random fields models. In fact random fields models accommodate both finite and infinite-factor models and thus all finite-factor models are special cases of random fields models. It can be shown that the Gaussian random fields can be interpreted as a linear combination of infinite number of Browian motions indexed by different forward rate maturities, $T$, under mild technical conditions. Thus random fields models can be viewed as an infinite-factor generalization of HJM models. Pang [60] shows that it is possible to reduce Eq.(2.6) to a $d$-factor HJM model by taking

$$d\bar{W}(t, T) = \frac{1}{\sigma(t, T)} \sum_{i=1}^{d} \sigma_i(t, T) d\bar{W}_i(t).$$

(2.8)

In the HJM framework (see Sec.1.4), the covariance of instantaneous changes of forward rate is given as

$$\text{Cov}[df(t, u), df(t, v)] = \sigma(t, u) \cdot \sigma(t, v),$$

(2.9)

where $\sigma(t, u)$ is a $d$-dimension vector and $\cdot$ is the inner product of two vectors. While for random fields model, we have

$$\text{Cov}[df(t, u), df(t, v)] = \sigma(t, u) \sigma(t, v)c(u, v).$$

(2.10)
If we rewrite the HJM volatilities $\sigma(t, u)$ in Eq.(1.11) as

$$\sigma(t, T) = \sigma(t, T)e(t, T),$$

(2.11)

with $|e(t, T)| = 1$, the Gaussian field model and HJM model are equivalent if

$$c(u, v) = e(t, u) \cdot e(t, v).$$

(2.12)

This shows the key difference between the calibration of HJM and random fields models. We can directly specify the correlation structure without the determination of the number of factors. Thus we know that correlation structure are more important and the number of factors is only used to reproduce empirical or implied correlation.

Second, re-calibration is unnecessary for random fields models. Although the no-arbitrage models are consistent with the current term structure, they require the re-calibration of the parameters of the models in order to fit the new current term structure. As pointed out in Buraschi and Corielli [15], in the HJM framework, the finite-factor models allow only one degree of freedom: the choice of one yield curve initial condition at initial time. From this, the model completely determines the future trend of the yield curve and thus its possible shapes in the future. For instance, the $d$-factor models can at most fit $d$ points on the yield curve and the volatility curve at a specific date. At the following date, without re-calibration, the same specification will typically miss those points.

In practice, the most common solution is to run a continuous re-calibration by inputting the new initial term structure. However, as Pang [60] pointed out, the re-initialization of yield curve at each new date would violate the no-arbitrage constraint, which is the assumption of the no-arbitrage modeling. In other words, the re-calibration violates the self-financing condition of the replication strategy since it implies a change in the conditional distribution of the process with respect to which the replicating portfolio weights are computed. In $d$-factor HJM term structure mod-
els, any security can be perfectly hedged by a preferred choice of \( d \) assets. However, in random fields HJM models, the innovation of each instantaneous forward is imperfectly correlated with that of any linear combination of other instantaneous forwards. Thus random fields models may have enough freedom to fit the current yield curves. More discussion of re-calibration can be found in Pang [60], which showed that in the interest rate case the calibration of a random fields model in comparison to a \( d \)-factor HJM model permits more stability over time and frequent re-calibration can be avoided. Pang [60] examined the stability of the covariance function and showed that the function maintained similar shapes throughout a long period of time (at least one month). The eigenvalues and corresponding eigenfunctions of the implied zero rate covariance matrix are almost similar during the period. He also examined the HJM models, which usually use principle component analysis (PCA) to extract the eigenvalues and eigenvectors. He found that the eigenvalues and eigenvectors in HJM models are very unstable during different periods.

There are many other advantages for modeling the term structure as random fields. For example, the fact that volatility estimation is imperfect can be shown in random fields interest rate modeling. See Goldstein [26] for more details.

2.3 Random Fields LIBOR Market Model (RFLMM)

In this section, we derive the dynamics of LIBOR rates \( L_k(t) \) with uncertainty terms modeled as random fields, under risk neutral measure \( \mathbb{Q} \), and \( T_j \)-forward measure \( \mathbb{Q}_{T_j} \), for \( j = 0, 1, ..., N \).

Let us consider the time structures \( \{T_0, T_1, ..., T_N\} \) with time intervals \( \delta_k = T_k - T_{k-1}, k = 1, ..., N \). For \( t < T_{k-1} < T_k \), the LIBOR forward rate \( L_k(t) \) is defined in Eq.(1.12) and the zero coupon bond price \( P(t, T) \) is defined in Eq.(1.2). Given the dynamics of \( f(t, T) \) in Eq.(2.6), by It\(\hat{o}\)’s formula we can derive that the dynamics of
the zero coupon bond price $P(t, T)$ with random field setting is
\[
dP(t, T) = P(t, T)[r(t)dt - \int_t^T \sigma(t, u)d\tilde{W}(t, u)du].
\] (2.13)

The dynamics of $L_k(t)$ is determined by those of zero coupon bonds. By Itô’s formula we can derive that under risk neutral measure $Q$, the dynamics of $L_k(t)$ is:
\[
dL_k(t) = \frac{1}{\delta_k} \frac{P(t, T_{k-1})}{P(t, T_k)} \left[ \int_{t}^{T_k} \sigma(t, u)d\tilde{W}(t, u)du + \int_{t}^{T_k} \sigma(t, u)d\tilde{W}(t, u)du \right].
\] (2.14)

Now let us derive the dynamic of forward rates under $T_k$-forward measure. Suppose that there exists a function $\theta(t, T_k, u)$ such that $dW^{T_k}(t, u) := \theta(t, T_k, u)dt + d\tilde{W}(t, u)$ has normal distribution $\Phi(0, dt)$ under $T_k$-forward measure. Using the fact that if $L_k(t)$ is a martingale under $T_k$-forward measure then the drift term should vanish, we can conclude that
\[
\theta(t, T_k, u) = \int_{t}^{T_k} \sigma(t, v)c(u, v)dv.
\] (2.15)

Thus we have the dynamics of $T_k$ under $T_k$-forward measure as shown in the following theorem and corollaries.

**Theorem 2.3.1. (Random fields dynamics under the associated forward measures)** Under the lognormal assumptions, the dynamics of the instantaneous forward rates $L_k(t)$ under the $T_k$-forward measure is described by the following equation
\[
dL_k(t) = \frac{1}{\delta_k} \frac{P(t, T_{k-1})}{P(t, T_k)} \int_{T_{k-1}}^{T_k} \sigma(t, u)dW^{T_k}(t, u)du,
\] (2.16)

where $W^{T_k}(t, u)$ is a Gaussian random field under $T_k$-forward measure.

See Appendix B.1 for the proof of Theorem 2.3.1.

**Corollary 2.3.2.** If $dW(t, u)$ has normal distribution $\mathcal{N}(0, dt)$ under the risk neutral measure, then
\[
dW^{T_k}(t, u) := \int_{t}^{T_k} \sigma(t, v)c(u, v)dvdt + d\tilde{W}(t, u)
\]
has normal distribution \( N(0, dt) \) under \( T_k \)-forward measure (using \( P(t, T_k) \) as a numeraire).

**Corollary 2.3.3.** If \( \text{corr}[dW^{T_k}(t, u), dW^{T_k}(t, v)] = c(u, v) \), then

\[
\int_{T_k}^{T_{k-1}} \int_{T_k}^{T_{k-1}} \sigma(t, u)\sigma(t, v) c(x, y) dx dy dt
\]

has normal distribution with mean 0 and variance under \( T_k \)-forward measure.

Theorem 2.3.1 shows that \( L_k(t) \) also has lognormal type distribution under \( T_k \)-forward measure. Corollary 2.3.2 describes the form of random fields under \( T_k \)-forward measure and Corollary 2.3.3 provides the distribution of the integral of random fields, given the correlation structure \( c(u, v) \). These results are essential for the derivation of random fields LIBOR market model.

Now we can derive the dynamics of forward rates \( L_k(t) \) under \( T_j \)-forward measure, \( j = 1, 2, \ldots, N \). From Eq.(2.16) and Corollary 2.3.2, we can derive the relation of \( dW^{T_k}(t, u) \) and \( dW^{T_{k+1}}(t, u) \) as follows,

\[
dW^{T_k}(t, u) = dW^{T_j}(t, u) - \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \sigma(t, v) c(u, v) dv dt,
\]

for \( j > k \). As we know, \( L_k(t) \) is a martingale under \( T_k \)-forward measure. By Martingale representation theorem under random field, there exists a function \( \xi(t, u) \), such that

\[
dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_k}(t, u) du.
\]

Comparing the above equation with Eq.(2.16), we can simply take

\[
\xi_k(t, u) = \frac{\delta_L L_k(t) + 1}{\delta_L L_k(t)} \sigma(t, u),
\]

and get the dynamics of \( L_k(s) \) under \( T_k \)-forward measure. The derivation in case \( j < k \) is similar thus we have the following theorem.

**Theorem 2.3.4. (Random fields dynamics under forward measures)** Under the lognormal assumptions, the dynamics of the instantaneous forward rates \( L_k(t) \)
under the $T_k$-forward measure, in three cases $j < k, j = k, j > k$, are described respectively by the following equations

\[
\begin{cases}
    dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) [dW^{T_j}(t, u) + \Lambda_j^k(t, u)dt]du, & j < k; \\
    dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)dW^{T_j}(t, u)du, & j = k; \\
    dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^{T_j}(t, u) - \Lambda_j^k(t, u)dt]du, & j > k.
\end{cases}
\]

with

\[
\Lambda_j^k(t, u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \delta_i L_i(t) \xi_i(t, v)c(u, v) \delta_i L_i(t) + 1 dv,
\]

where $W^{T_j}(t, u)$ is a Gaussian random field under $T_j$-forward measure. The above equations admit a unique solution if the coefficient $\xi_k(\cdot, \cdot)$ are locally bounded, locally Lipschitz continuous and predictable.

See Appendix B.2 for the proof of Theorem 2.3.4.

**Theorem 2.3.5. (Consistency of LMM and RFLMM)** HJM framework is a special case of Kennedy-Goldstein framework, since Theorem 2.3.4 will reduce to Eq.(2.6), if we take $\delta_i \to 0$. LMM is a discrete case of RFLMM, since Theorem 2.3.4 will reduce to Eq.(1.18), if we take the value of $dW^{T}(t, u)$ on $[T_{k-1}, T_k]$ to be $dW^{T_k}(t)$.

See Appendix B.3 for the proof of Theorem 2.3.5.

Theorem 2.3.5 shows that LIBOR market model Eq.(1.18) is a discrete case of random field LIBOR market model, as we have declared in Sec.1.4.

### 2.4 Option Pricing in Random Fields LIBOR Market Model

In this section we derive the Black-Scholes equation for the pricing of derivatives in random field case and thus the closed-form formulas for caplets and swaptions.
Analogous to the Black-Scholes equation for the price of derivatives in Brownian motion case, we provide and prove a Black-Scholes type equation in random fields setting, as shown in the following theorem. The proof of the theorem is shown in Appendix C.

**Theorem 2.4.1. (The Black-Scholes equation with random field setting for time dependent parameters)** Suppose we have an option on some underlying asset $S$, which has dynamics

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \int_{t_1}^{t_2} \xi(t, u)dW(t, u)du,$$

with time dependent parameters. Then the price of derivatives $V$ follows the equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left[ \int_{t_1}^{t_2} \int_{t_1}^{t_2} \xi(t, u)\xi(t, v)c(u, v)dudv \right] \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - r(t)V = 0.$$  \hspace{1cm} (2.22)

We can denote it as Black-Scholes equation for option pricing with random field. For a call option with strike $K$ and maturity $T$, the price is

$$C = SN(\tilde{d}_1) - Ke^{-\int_0^T r(u)du}N(\tilde{d}_2),$$

where

$$\tilde{d}_1 = \frac{1}{\sqrt{\int_0^T \hat{\xi}^2(u, t_1, t_2)du}} [\ln \frac{S}{K} + \int_0^T (r(u) + \frac{\hat{\xi}^2(u, t_1, t_2)}{2})du];$$

$$\tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_0^T \hat{\xi}^2(u, t_1, t_2)du};$$

$$\hat{\xi}^2(t, t_1, t_2) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \xi(t, u)\xi(t, v)c(u, v)dudv.$$

We can get a Black-type formula for pricing of caplets and swaptions from Theorem 2.4.1, which means that the Black’s formula with random field is similar to that derived by Black [11], as shown in Eq.(1.23). Thus we can derive the pricing formulas for caplets and swaptions.
2.4.1 Random fields LMM formula for caplets. The payoff of a caplet at time $T_k$ is

$$\delta_k [L_k(T_{k-1}) - K]^+. $$

The time $t$ price of caplet is

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E_T \left[ [L_k(T_{k-1}) - K]^+ | \mathcal{F}_t \right].$$

Suppose the LIBOR rate $L_k(t)$ has dynamics as Eq.(2.18) under $T_k$-forward measure. If we assume that $\xi_k(t, u)$ is deterministic, then by Corollary 2.3.3, $L_k(t)$ has normal distribution and by Black’s formula the time $t$ price is

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \text{Black}(K, L_k(t), \sigma_{RF}^{\text{Black}} \sqrt{T_{k-1} - t}),$$

where

$$\sigma_{RF}^{\text{Black}} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \left[ \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} \xi_k(s, x) \xi_k(s, y) c(x, y) dxdy \right] ds.} \quad (2.23)$$

We can specify the volatility and correlation structure to get more particular form of the price. For example, Kennedy [47] demonstrated that for forward rate to be modeled as a Gaussian, Markov, and stationary random field, the volatility must be of the form $\xi(t, T) = \sigma e^{-\alpha(T-t)}$ and the correlation structure must be $\text{Corr}[dW(t, T_1), dW(t, T_2)] = e^{-\rho |T_2 - T_1|}$. In this case, the caplet price under random field LMM is given in the above formula, where

$$(\sigma_k^{\text{Black,RF}})^2 = \sigma^2 e^{2t} \left[ \frac{\rho - \alpha}{\alpha^2 - \rho^2} (e^{-2\alpha T_k} - e^{-2\alpha T_{k-1}}) - \frac{\delta}{\alpha + \rho} e^{-(\alpha + \rho) T_k} + \frac{\delta}{\alpha - \rho} e^{-(\alpha - \rho) T_{k-1}} \right].$$

2.4.2 Random fields LMM formula for swaptions. The payoff of a swaption at time $T_k$ is

$$\delta_k [S_{i,j}(T_{k-1}) - K]^+. $$

The time $t$ price is therefore

$$[S_{i,j}(t) - K]^+ \sum_{k=i+1}^{j} \delta_k P(t, T_k).$$
By the same discussion in Sec.1.5.2, we can derive that the uncertainty term of $S_{i,j}(t)$ in random field LMM. And by Black’s formula the time $t$ price of swaption is

$$\text{Swpt}(t, K, T_i, T_j) = A \text{ Black}(K, S_{i,j}(t), \sigma_{i,j}^{\text{Black,RF}} \sqrt{T_{k-1} - t}),$$

where

$$\sigma_{i,j}^{\text{Black,RF}} = \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \left[ \sum_{k=i+1}^{j} \frac{\delta_k L_k(s) \gamma_k^{i,j}(s)}{1 + \delta_k L_k(s)} \int_{T_{k-1}}^{T_k} \xi_k(s) dW(s, u) \right]^2 ds}$$

$$= \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)}}$$

$$\times \int_t^{T_i} \int_t^{T_k} \int_t^{T_l} \int_t^{T_{k-1}} \xi_k(s, x) \xi_l(s, y) c(x, y) dx dy ds. \quad (2.24)$$

The last equation is obtained by using standard freezing approximation techniques, i.e. approximatively evaluating the LIBOR rates $L_k(s), t \leq s \leq T_i$ appearing in the instantaneous volatility at initial time $t$.

In this chapter we have derived our first new LIBOR market model, the random fields LIBOR market model (RFLMM) (Eq.(2.20)) and the implied volatility formulas for caplets (Eq.(2.23)) and swaptions (Eq.(2.24)). We can use these two formulas to calibrate the RFLMM, with appropriate choice of term structure curves.
CHAPTER 3  
RANDOM FIELDS VOLATILITY SMILE MODELS

In this chapter, we derive the random fields volatility models. First, we review the phenomena of volatility smiles in interest rates modeling in Sec.3.1. Second, we extend the lognormal mixture model to random fields case in Sec.3.2, as an example of random fields local volatility models. Third, we derive two random fields stochastic volatility models in Sec.3.3, for SABR type and Wu-Zhang type respectively.

3.1 Volatility Smile

It is well known that the lognormal LMM has the main drawback of producing constant implied volatility for any given maturity. From the Black’s formula for caplet price Eq.(1.22), we can see that the volatility of the forward rate does not depend on the option strike $K$. However, each caplet market price requires its own Black volatility depending on the caplet strike $K$. In other terms, there is not a single volatility parameter $\sigma_k$ such that both

$$C_{\text{plt}}(t, K_1, T_{k-1}, T_k) = \delta_k P(t, T_k) \text{Black}(K_1, L_k(t), \sigma_k \sqrt{T_k - t})$$

and

$$C_{\text{plt}}(t, K_2, T_{k-1}, T_k) = \delta_k P(t, T_k) \text{Black}(K_2, L_k(t), \sigma_k \sqrt{T_k - t})$$

hold. The market observation shows that we need two different volatilities $\sigma_k(K_1)$ and $\sigma_k(K_2)$ to use Black’s formula to match market price. The volatility smile or skew of the $T_k$-expiry caplet is the curve $K \rightarrow \sigma_k(K)/\sqrt{T_k - t}$. The reason that the curve is called volatility ‘smile’ or ‘skew’ is that the curve usually displays ‘smiley’ or ‘skewed’ shapes. If the volatility has a lower value around the at-the-money derivatives the shape is called smile. If the low-strike implied volatilities are higher than high-strike implied volatilities, the shape is said to be skew.
The model’s volatility smile is generated as follows. Given a starting strike $K$, we compute the caplet price using

$$C_{\text{plt}}(K) = \delta_k P(t, T) E^{T_k}[L_k(T_{k-1}) - K]^+ |\mathcal{F}_t],$$

with $L_k(t)$ following dynamics Eq.(3.1). Then we invert Black’s formula for this strike, i.e., solve

$$C_{\text{plt}}(K) = \delta_k P(t, T) \text{Black}(K, L_k(t), \sigma_k(K) \sqrt{T_k - t})$$

in $\sigma_k(K)$. Finally we change $K$ and repeat the process to get a curve $K \rightarrow \sigma_k(K)$.

There have been many works dealing with the volatility smiles. One popular adjustment to the above issue is to start from an alternative dynamics, by assuming that under $T_k$–forward measure,

$$dL_k(t) = \xi_k(t, L_k(t))L_k(t)dW_{T_k}(t), \quad (3.1)$$

where $\xi_k$ can be either a deterministic or a stochastic function of $L_k(t)$. A deterministic $\xi_k$ leads to so called ‘local-volatility model’. For instance, we can take $\xi_k(t, L_k) = \xi_k(t)L_k(t)^{\beta-1}$, where $0 \leq \beta \leq 1$ and $\xi_k(t)$ is a deterministic function of $t$. The latter case leads to so called ‘stochastic volatility models’. For example, we can take $\xi_k(t, L) = \xi_k(t)$, where $\xi_k(t)$ follows a stochastic differential equation.

Brigo and Mercurio [13] briefly review the major approaches proposed in the existing literature, including local volatility models (LVMs), jump diffusion models (JDMs), Levy driven models (LDMs), uncertain parameters models (UPMs) and stochastic volatility models (SVMs). Two popular extensions of the LMM are based on modeling stochastic volatility as in Heston [34] and Hagan et al.[27]. The first approach includes those of Hull and White [37] and Heston [34], with the related application to the LIBOR market model developed by Andersen and Andreasen [5], Piterbarg [62], Wu and Zhang [70]. The second approach follows Hagan’s method, including those of Hagan and Lesniewski [28], Henry-labordere [33], and Rebonato [65], commonly
referred to as Stochastic Alpha-Beta-Rho (SABR) model. We will use the lognormal-mixture model introduced by Brigo et al. [14] as an example of local volatility models and use SABR model and Wu-Zhang model [70] as examples of stochastic volatility models to derive the volatility smile models with random field case.

### 3.2 Random Fields Local Volatility Models

In this section we use a local volatility model to describe the volatility smile. We extend the lognormal mixture model derived in Brigo et al. [14] to the random field case. Analogous to Eq. (3.1), we can assume that under $T_k$-forward measure, the dynamics of $L_k(t)$ with random field is

$$dL_k(t) = \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW_{T_k}(t, u) du. \quad (3.2)$$

Analogous to LIBOR market model, we have the following theorem.

**Theorem 3.2.1.** (Random field local volatility dynamics under forward measures) Under the lognormal assumptions, the dynamics of the instantaneous forward rates $L_k(t)$ under the $T_j$-forward measure, in three cases $j < k$, $j = k$, $j > k$, are described respectively by the following equations

$$\begin{align*}
\frac{dL_k(t)}{L_k(t)} &= \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) [dW_{T_j}(t, u) + \Lambda_j^k(t, u) dt] du, \quad j < k; \\
\frac{dL_k(t)}{L_k(t)} &= \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW_{T_j}(t, u) du, \quad j = k; \\
\frac{dL_k(t)}{L_k(t)} &= \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) [dW_{T_j}(t, u) + \Lambda_j^k(t, u) dt] du, \quad j > k.
\end{align*} \quad (3.3)$$

$$\Lambda_j^k(t, u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, L_k(t), v)c(u, v)}{\delta_i L_i(t) + 1} dv,$$

where $W_{T_j}(t, u)$ is a Gaussian random field under $T_j$-forward measure. The above equations admit a unique solution if the coefficients $\xi_k(\cdot, \cdot, \cdot)$ are locally bounded, locally Lipschitz continuous and predictable.
Brigo and Mercurio [13] provide a class of analytical models based on a given mixture of densities. In this section we will extend them to the random fields setting and provide the closed-form formula for caplet pricing and corresponding implied volatility that can be used for calibration, based on the lognormal distribution assumptions.

Let us consider the diffusion process with dynamics

\[ dG^k_i(t) = G^k_i(t) \int_{T_{k-1}}^{T_k} \nu^k_i(t, G^k_i(t), u) dW^T_k(t, u) du, \quad (3.4) \]

with initial value \( G^k_i(0) = L^k_i(0) \) for all \( i = 1, 2, ..., M \), where \( W^T_k(t, u) \) is a Gaussian field under \( T_k \)-forward measure with correlation \( c(t, T_1, T_2) \) as described in Eq.(2.7).

Analogous to Brigo and Mercurio [13], the problem here is to derive the local volatility \( \xi^k_i(t, u) \) such that the density of \( L^k_i(t) \) under \( T_k \)-forward measure satisfies, for each time \( t \):

\[ p^k_i(x, t) = \frac{d}{dx} P^{T_k} \{ L^k_i(t) \leq x \} = \sum_{i=1}^{M} \omega_i \frac{d}{dx} P^{T_k} \{ G^k_i(t) \leq x \} du = \sum_{i=1}^{M} \omega_i p^k_i(x, t), \]

where \( \omega_i \) is a weight function with \( \sum_{i=1}^{M} \omega_i = 1 \). In fact \( p^k_i(x) \) is a proper density function under \( T_k \)-forward measure since

\[ \int_{0}^{+\infty} x p^k_i(x, t) dx = \int_{0}^{+\infty} x \sum_{i=1}^{M} \omega_i p^k_i(x, t) dx = \sum_{i=1}^{M} \omega_i G^k_i(0) = L^k_i(0), \]

if the conditions for exchange of integrals are verified. The last calculation comes from the fact that \( G^k_i(t) \) is a martingale under \( T_k \)-forward measure. We know that that the local volatility \( \xi^k_i(t, L^k_i(t)) \) is

\[ \int_{T_{k-1}}^{T_k} \xi^k_i(t, u) \xi^k_i(t, v) c(u, v) dvdu = \frac{\sum_{i=1}^{M} \omega_i \int_{T_{k-1}}^{T_k} \nu^k_i(t, u) \nu^k_i(t, v) c(u, v) dvdu p^k_i(x, t)}{\sum_{i=1}^{M} \omega_i p^k_i(x, t)}. \quad (3.5) \]
If we take $c(x, y) = 1$, the above formula reduces to

$$
\xi_k(t) = \sqrt{\sum_{i=1}^{M} \omega_i \nu^k_i(t)^2 p^k_i(x, t) - \sum_{i=1}^{N} \omega_i p^k_i(x, t)},
$$

which is the original lognormal-mixture model derived in Brigo et al. [14].

Since

$$
\int_{T_k}^{T_k} \xi_k(t, L_k(t), u) dW^{T_k}(t, u) du
$$

has normal distribution with variance

$$
\int_{T_k}^{T_k} \xi_k(t, L_k(t), u) \xi_k(t, L_k(t), x) c(u, v) dv du,
$$

we have the following theorem.

**Theorem 3.2.2. (Random fields lognormal-mixture model dynamics under forward measures)** The dynamics of LIBOR rate $L_k(t)$ is

$$
dL_k(t) = L_k(t) \sqrt{\sum_{i=1}^{M} \omega_i (\int_{T_k}^{T_k} \nu^k_i(t, u) \nu^k_i(t, v) c(u, v) dv du) p^k_i(x, t) - \sum_{i=1}^{N} \omega_i p^k_i(x, t)} dW^{T_k}(t),
$$

where $W^{T_k}(t)$ is a Brownian motion under $T_k$-forward measure.

If we assume that in Eq.(3.4)

$$
\nu^k_i(t, x, u) = \nu^k_i(t, u),
$$

i.e., the densities $p^k_i(x, t)$ are all lognormal, where for all $k$, $\nu^k_i(t)$ are deterministic and continuous functions of time that are bounded from above and below by strictly positive constants, the marginal density of $G^k_i(t)$ is then lognormal:

$$
p^k_i(x, t) = \frac{1}{x \nu^k_i(t) \sqrt{2\pi}} exp\left\{ -\frac{1}{2 \nu^k_i(t)^2} \ln \frac{x}{L_k(t)} + \frac{1}{2} \nu^k_i(t)^2 \right\},
$$

$$
\nu^k_i(t) = \sqrt{\int_{0}^{t} \int_{T_k}^{T_k} \nu^k_i(s, u) \nu^k_i(s, v) c(s, u, v) du dv ds}.
$$
3.2.1 Option pricing in a random fields local volatility model. In this section, we derive the closed-form pricing formulas of caplets in random field lognormal mixture model.

The payoff of a caplet at time $T_k$ is

$$
\delta_k [L_k(T_{k-1}) - K]^+.
$$

The time $t$ price of caplet is

$$
C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \mathbb{E}_{T_k}[L_k(T_{k-1}) - K]^+ | \mathcal{F}_t]
$$

$$
= P(0, T_k) \int_0^{+\infty} [x - K]^+ p_k(x, t) dx
$$

$$
= P(0, T_k) \sum_{i=1}^M \int_0^{+\infty} [x - K]^+ p_k^i(x, t) dx. \quad (3.11)
$$

Suppose that the LIBOR rate $L_k(t)$ has dynamics as Eq.(3.2) under $T_k$-forward measure and Eq.(3.8) holds, then the caplet price is

$$
C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \sum_{i=1}^M \text{Black}(K, L_k(t), \nu^k_i(T_{k-1}, u)).
$$

Given the above analytical tractability, we can derive an explicit approximation for the caplet implied volatility as a function of the caplet strike price.

**Theorem 3.2.3. (Implied volatility of random fields lognormal-mixture model)** Define $m = \ln \frac{L_k(t)}{K}$. The implied volatility $\sigma^{Black,MRF}_k$ is

$$
\sigma^{Black,MRF}_k(m) = \sigma^{Black,MRF}_k(0) + \frac{1}{2\sigma^{Black,MRF}_k(0)(T_k - 1)} \sum_{i=1}^M \omega_i \left[ \frac{\sigma^{Black,MRF}_k(0) \sqrt{T_k - 1} - t}{V_k^i(T_k - 1)} \right] e^{\frac{1}{2}(\sigma^{Black,MRF}_k(0))^2(T_k - 1) - \nu^k_i(T_{k-1})^2} - 1]m^2 + o(m^2), \quad (3.12)
$$

where the at-the-money implied caplet volatility $\sigma^{Black,MRF}_k(0)$ is

$$
\sigma^{Black,MRF}_k(0) = \frac{2}{\sqrt{T_k - 1 - t}} \Phi^{-1} \left( \sum_{i=1}^M \omega_i \Phi(\frac{1}{2}\nu^k_i(T_{k-1})) \right). \quad (3.13)
$$
Theorem 3.2.3 can be used to capture the volatility smiles in the calibration of random field lognormal-mixture model to caplets.

3.3 Random Fields Stochastic Volatility Models

In this section, we derive stochastic volatility models when interest rates are described as random fields, following the approaches by Heston [34] and Hagan et al. [27]. The following dynamics assumes the evolutions of forward rates $L_k(t)$ under the associated $T_k$-forward measure $Q^T_k$ and also specifies the evolution of volatility process under LIBOR spot measure $Q^T$:

$$
\begin{align*}
\begin{cases}
    dL_k(t) = L_k(t)\xi_k(t,u)\text{d}W^T_k(t), \\
    dV_k(t) = a_k(t,V_k)dt + b_k(t,V_k)dB^T_k(t)
\end{cases}
\end{align*}
$$

where $W(t)$ and $B(t)$ are correlated. This dynamics is general enough to include both approaches mentioned above, after some appropriate adjustments. In random fields case, it is natural to assume the dynamics of forward rates under $T_k$-forward measure and the dynamics of volatility process under LIBOR spot measure as:

$$
\begin{align*}
\begin{cases}
    dL_k(t) = L_k(t)\int_{T_{k-1}}^{T_k} \xi_k(t,u)\text{d}W^T_k(t,u)du, \\
    dV_k(t,u) = a_k(t,u)dt + b_k\text{d}W^T(t,u)
\end{cases}
\end{align*}
$$

(3.14)

with

$$
\text{corr}[\text{d}W^T_k(t,1),\text{d}W^T_k(t,2)] = c(t,T_1,T_2),
$$

where $\xi_k(t,u) = \xi_k(t,u,L_k(t),V(t,u))$; $a(t,u) = a(t,u,V(t,u))$; $b(t,u) = b(t,u,V(t,u))$ and $W^T_k(t,u)$, $W^T(t,u)$ are random fields under $T_k$-forward measure $Q^T_k$ and LIBOR spot measure $Q^T$. Since $\text{d}W^T_k(t,u)\text{d}W^T_k(t,v) = c(u,v)dt$, we have that

$$
\int_{T_{k-1}}^{T_k} \xi_i(t,v)\text{d}W^T_k(t,v)\text{d}W^T_k(t,u)dv = \int_{T_{k-1}}^{T_k} \xi_i(t,v)c(u,v)dvdt. 
$$

Analogous to the deviation in Sec.2.3, we have the following theorem.
Theorem 3.3.1. (Random fields stochastic volatility dynamics under forward measures) Under the lognormal assumptions, the dynamics of the instantaneous forward rates $L_k(t)$ and $V(t, u)$ under the $T_j$-forward measure, in three cases $j < k, j = k, j > k$, are described respectively by the following equations

$$
\begin{align*}
\begin{cases}
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^T_j(t, u) + \Lambda_j^k(t, u)dt]du, \\
    dV_k(t, u) &= a_k(t, u)dt + b_k(t, u)[dW^T_j(t, u) + \Lambda_j^k(t, u)dt], \quad j < k; \\
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)dW^T_j(t, u)du, \\
    dV_k(t, u) &= a_k(t, u)dt + b_k(t, u)[dW^T_j(t, u) + \Lambda_j^k(t, u)dt], \quad j = k; \\
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^T_j(t, u) + \Lambda_j^k(t, u)dt]du, \\
    dV_k(t, u) &= a_k(t, u)dt + b_k(t, u)[dW^T_j(t, u) + \Lambda_j^k(t, u)dt], \quad j > k;
\end{cases}
\end{align*}
\tag{3.15}$$

with

$$\Lambda_j^k(t, u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, v)c(u, v)}{\delta_i L_i(t) + 1}dv,$$

where $W^T_j(t, u)$ is a Gaussian random field under $T_j$-forward measure. These equations admit a unique solution if the coefficients $\xi_k(\cdot, \cdot, \cdot)$ are locally bounded, locally Lipschitz continuous and predictable.

The dynamics in Theorem 3.3.1 are general enough to include all the stochastic volatility models we mentioned above. For example, we can derive the stochastic volatility dynamics with random fields setting following the SABR approach:

$$
\begin{align*}
\begin{cases}
    dL_k(t) &= L_k(t)\beta_k \int_{T_{k-1}}^{T_k} V_k(t, u)dW^T_k(t, u)du, \\
    dV_k(t, u) &= \alpha_k(t)V_k(t, u)dW^T_k(t, u)
\end{cases}
\end{align*}
$$

which is according to choose

$$\xi_k(t) = V_k(t, u)L_k(t)^{\beta_k-1}, \quad a_k(t) = 0, \quad \text{and} \quad b_k(t) = \alpha_k(t)V_k(t, u)$$

in Eq.(3.15), where
\(\beta_k > 1, \alpha_k, \gamma_k\) are positive constants.

And we can derive the stochastic volatility dynamics with random field setting following the Wu and Zhang [70] approach:

\[
\begin{align*}
    dL_k(t) &= \alpha_k L_k(t) \int_{T_{k-1}}^{T_k} \sqrt{V_k(t,u)} dW_k(t,u) du, \\
    dV_k(t,u) &= \beta_k (\theta_k - V_k(t,u)) dt + \gamma_k \sqrt{V_k(t,u)} dW_k(t,u),
\end{align*}
\]

which is according to choose

\[
    \xi_k(t) = \alpha_k \sqrt{V_k(t,u)}, \quad a_k(t) = \beta_k (\theta_k - V_k(t,u)), \quad b_k(t) = \gamma_k \sqrt{V_k(t,u)}
\]

in Eq.(3.15), where \(\beta_k, \alpha_k, \theta_k, \gamma_k\) are positive constants.

### 3.3.1 SABR approach.

The SABR approach derived in Hagan et al. [27] is an extension of the Constant Elasticity of Variance model (CEV), which models the dynamics of a single forward rate as

\[
dL(t) = \sigma L(t)^\beta dW(t).
\]

The full dynamics of the SABR model is

\[
\begin{align*}
    dL(t) &= V(t)L(t)^\beta dW(t) \\
    dV(t) &= \alpha(t)V(t)dZ(t)
\end{align*}
\]

with two correlated Brownian motion \(W(t)\) and \(Z(t)\). Thus it is natural to derive the random field SABR/LMM model as

\[
\begin{align*}
    dL_k(t) &= L_k(t)^\beta \int_{T_{k-1}}^{T_k} V_k(t,u) dW_k(t,u) du \\
    dV(t,u) &= \alpha_k V_k(t,u) dW_k(t,u)
\end{align*}
\]

with correlation structure as before. Hagan and Lesniewski [28] derived the SABR/LMM model and approximate closed-form formulas for caplets and swaption pricing. In this section we will extend the model to random field case and derive the corresponding formulas for caplets and swaptions pricing.
As shown in Sec.3.3, the dynamics of $L_k(t)$ and $V_k(t, u)$ under the $T_j$-forward measure, in three cases $j < k, j = k, j > k$, are described respectively by the following equations

\[
\begin{align*}
    dL_k(t) &= L_k(t)^{\beta_k} \int^{T_k}_{T_{k-1}} V_k(t, u)[dW^T_j(t, u) + \Psi^j_k(t, u)dt]du \\
    dV_k(t, u) &= V_k(t, u)\alpha_k(t, u)[dW^T_j(t, u) + \Psi^k_j(t, u)dt] \quad j < k; \\
    dL_k(t) &= L_k(t)^{\beta_k} \int^{T_k}_{T_{k-1}} V(t, u)dW^T_j(t, u)du \\
    dV_k(t, u) &= \alpha_k(t, u)V_k(t, u)dW^T_j(t, u) \quad j = k; \\
    dL_k(t) &= L_k(t)^{\beta_k} \int^{T_k}_{T_{k-1}} V_k(t, u)[dW^T_j(t, u) + \Psi^j_k(t, u)dt]du \\
    dV_k(t, u) &= V_k(t, u)\alpha_k(t, u)[dW^T_j(t, u) + \Psi^j_k(t, u)dt] \quad j > k.
\end{align*}
\]  

(3.16)

with

\[
\Psi^j_k(t, u) = \sum_{i=I+1}^{j} \int^{T_i}_{T_{i-1}} \frac{\delta_i L_i(t)^{\beta_k} V_i(t, v)c(u, v)}{\delta_i L_i(t) + 1}dv,
\]

which is according to choose

$\xi_k(t) = V_k(t, u)L_k(t)^{\beta_k-1}$, $a_k(t) = 0$, and $b_k(t) = \alpha_k(t)V_k(t, u)$ in Eq.(3.15), where $\beta_k > 1$ are positive constants.

The above model does not have analytic closed-form solutions. However, there exits a strong solution under appropriate conditions on $\beta_i$’s and choice of boundary conditions. Here we will derive an approximate solution by stochastic Taylor expansions, which are introduced in details in Kloeden and Platen [48]. A version of stochastic Taylor expansions for our needs and its extension to the case of random fields model are presented in Appendix D. In the following we will derive the approximate solution for LIBOR rates $L_k(t)$ and variance processes $V_k(t)$ in the spirit of Milstein scheme in random fields setting.

From Eq.(3.16) we know that under the $T_j$-forward measure, the dynamics of
\( L_k(t) \) \((j < k)\) is

\[
\begin{cases}
  dL_k(t) = L_k(t)^{\beta_k} \int_{T_{k-1}}^{T_k} V_k(t, u)[dW^T_j(t, u) + \Psi^k_j(t, u)dt]du \\
  dV_k(t, u) = \alpha_k(t, u)V_k(t, u)[dW^T_j(t, u) + \Psi^k_j(t, u)dt]
\end{cases}
\]

(3.17)

Take

\[
\begin{align*}
  f_1 = & \ L_k(t)^{\beta_k} V_k(t, u)\Psi^k_j(t, u); \\
  g_1 = & \ L_k(t)^{\beta_k} V_k(t, u);
\end{align*}
\]

\[
\begin{align*}
  f_2 = & \ \alpha_k(t, u)V_k(t, u)\Psi^k_j(t, u); \\
  g_2 = & \ \alpha_k(t, u)V_k(t, u);
\end{align*}
\]

and apply Eq.(D.5) and Eq.(D.3) to \( L_k(t) \) and \( V_k(t, u) \) respectively, we have that

\[
\begin{align*}
  L_k(t_{n+1}) &= L_k(t_n) + L_k(t_n)^{\beta_k} \Psi^k_j(t_n, u)\Delta t_n + \sum_{t_1 \leq u \leq t_2} \delta_n L_k(t_n)^{\beta_k} V_k(t_n, u)\Delta W(t_n, u) \\
  &+ \sum_{t_1 \leq u \leq t_2} \delta_n KL_k(t_n)^{\beta_k} V_k(t_n, u)[\sum_{t_1 \leq s \leq t_2} \delta_n \frac{1}{2}(\Delta W(t_n, s)^2 - \Delta t_n)] + \sum_{t_1 \leq u \leq t_2} \delta_n \frac{1}{2}(\Delta W(t_n, s)^2 - \Delta t_n)],
\end{align*}
\]

\[
\begin{align*}
  V_k(t_{n+1}, u) &= V_k(t_n, u) + \alpha_k(t_n, u)V_k(t_n, u)\Psi^k_j(t_n, u)\Delta t_n + \alpha_k(t_n, u)V_k(t_n, u)\Delta W(t_n) \\
  &+ K\alpha_k(t_n, u)V_k(t_n, u)\frac{1}{2}(\Delta W(t_n)^2 - \Delta t_n).
\end{align*}
\]

Notice that there is one more term for \( L_k(t) \) which contains \( V_k(t, u) \) since \( V_k(t, u) \) is in the coefficients of \( L_k(t) \). In fact this approximation is very accurate. We will combine this Milstein scheme with Monte Carlo method in next chapter, which shows that Milstein scheme has almost the same accuracy and much less computing time than the usual Monte Carlo method. The caps/floors and swaptions prices can be calculated once \( L_k(t) \)'s are simulated by Monte Carlo methods.

### 3.3.2 Wu-Zhang approach

As an example, following Heston’s approach, we show how to price caplets and swaptions in Andersen and Brotherton-Ratcliffe model [6]. Andersen and Andreasen [12] extended the classical lognormal LMM to allow freely specifiable volatility, with arbitrage free dynamics as:
\[ dL_k(t) = \phi(L_k(t))\xi_k(t)dW^{T_k}(t), \]

where \( W^{T_k}(t) \) is a scalar Brownian motion under \( T_k \)-forward measure. Andersen and Brotherton-Ratchliffe [6], and Wu and Zhang [70] extended this set-up by specifying a scalar mean-reverting variance process of the form:

\[
\begin{align*}
  dL_k(t) &= \sigma_k L_k(t)\sqrt{V(t)}dW^{T_k}(t) \\
  dV(t) &= \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}dW^{T}(t)
\end{align*}
\]

(3.18)

where \( W^{T}(t) \) is a scalar Brownian motion under spot measure \( Q^T \) and \( dW^{T}(t)dW^{T_k}(t) = 0 \). This assumption of independence of source of uncertainty will leave the drift term of \( dV(t) \) unchanged under \( T_k \)-forward measure, which means that under \( T_k \)-forward measure \( Q^{T_k} \) the dynamics are:

\[
\begin{align*}
  dL_k(t) &= \sigma_k L_k(t)\sqrt{V(t)}dW^{T_k}(t) \\
  dV(t) &= \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}dW^{T_k}(t)
\end{align*}
\]

They developed asymptotic expansion for the pricing of these instruments. However, the assumption of independence of uncertainty may not reflect the skewness of volatility smile. One solution is to introduce another Brownian motion \( Z(t) \) for the variance processes, which are correlated to \( W(t) \). In the spirit of this idea we extend the model to random fields case as

\[
\begin{align*}
  dL_k(t) &= \sigma_k L_k(t)\int_{T_{k-1}}^{T_k} \sqrt{V(t,u)}dW^{T_k}(t,u)du \\
  dV(t,u) &= \kappa(\theta - V(t,u))dt + \epsilon \sqrt{V(t,u)}dW^{T}(t,u)
\end{align*}
\]

and

\[ corr(dW(t,u), dW(t,v)) = c(u, v), \]

where \( \sigma_k \) is positive constant and \( \kappa, \theta, \epsilon \) are positive constants. Since the measure change doesn’t change the correlation of Brownian motion, the above correlation can be in both measures, \( Q^{T_k} \) and \( Q^T \).
Now we derive the dynamics under $Q^T_k$ measure. The numeraire of spot LIBOR measure $Q^T$ associated to time structures $T = \{T_0, \ldots, T_N\}$ is the discretely rebalanced bank account $B^T$:

$$B^T(t) = \frac{P(t, T_{\Xi(t)-1})}{\prod_{j=0}^{\Xi(t)-1} P(T_{j-1}, T_j)},$$

where $\Xi(t) = i$ if $T_{i-2} < t < T_{i-1}$, $i \geq 1$.

The standard change of numeraire technique implies that when moving from measure $Q^T$ to measure $Q^T_k$, the drift term of a given process $X$ changes according to

$$\text{Drift}(X; Q^T_k) = \text{Drift}(X; Q^T) - \frac{d < X, \ln[B^T/P(\cdot, T_k)] >}{dt}.$$

Thus the dynamics of $L_k(t)$ and $V_k(t, u)$ under the $T_j$-forward measure, in three cases $j < k$, $j = k$, $j > k$, are described respectively by the following equations

$$\left\{ \begin{aligned}
\frac{dL_k(t)}{L_k(t)} &= \sigma_k \int_{T_{k-1}}^{T_k} \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon_{j+1}^k(t, u) dt] du, \\
\frac{dV(t, u)}{V(t, u)} &= \kappa(\theta - V(t, u)) dt - \epsilon \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon_{\Xi}^k(t, u) dt], \; j < k; \\
\frac{dL_k(t)}{L_k(t)} &= \sigma_k \int_{T_{k-1}}^{T_k} \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon_{k+1}^j(t, u) dt] du, \\
\frac{dV(t, u)}{V(t, u)} &= \kappa(\theta - V(t, u)) dt - \epsilon \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon_{\Xi}^j(t, u) dt], \; j > k;
\end{aligned} \right.$$  

(3.19)

with

$$\Upsilon_{\Xi}^j(t, u) = \sum_{i=\Xi(t)}^{j} \int_{T_{i-1}}^{T_i} \frac{\delta_i \sigma_i L_i(t) \sqrt{V(t, u)} c(u, v)}{1 + \delta_i L_i(t)} dv.$$  

Remark 3.3.2. An exact formula can be obtained by simply setting the correlation of Brownian fields to zero, since in this case the drift term will not change as a
consequence of the measure change. In fact, the correlation can be arbitrary and used for calibration of market caplets skew.

From Eq. (3.19) we know that under $T_j$-forward measure the dynamics of $L_k(t)$ and $V(t, u)$ for $j < k$ is

$$
\begin{align*}
    dL_k(t) &= \sigma_k L_k(t) \int_{T_{k-1}}^{T_k} \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon^k_j(t, u)] du \\
    dV(t, u) &= \kappa(\theta - V(t, u)) dt - \epsilon \sqrt{V(t, u)} [dW^T_j(t, u) + \Upsilon^k_T(t, u)] dt
\end{align*}
$$

Take

$$
\begin{align*}
    f_1 &= \sigma_k L_k(t) \sqrt{V(t, u)} \Upsilon^k_j(t, u); \\
    g_1 &= \sigma_k L_k(t) \sqrt{V(t, u)}; \\
    f_2 &= \kappa(\theta - V(t, u)) - \epsilon \sqrt{V(t, u)} \Upsilon^k_T(t, u); \\
    g_2 &= -\epsilon \sqrt{V(t, u)};
\end{align*}
$$

and apply (D.5) and (D.3) to $L_k(t)$ and $V(t, u)$ respectively, we have that

$$
\begin{align*}
    L_k(t_{n+1}) &= L_k(t_n) + \sigma_k L_k(t_n) \sqrt{V(t_n, u)} \Upsilon^k_j(t_n, u) \Delta t_n + \sum_{t_1 \leq u \leq t_2} \delta_n \sigma_k L_k(t_n) \sqrt{V(t, u)} \\
    &\quad - \sum_{t_1 \leq u \leq t_2} \delta_n \frac{1}{2} \kappa L_k(t_n) \sqrt{V(t_n, u)} \epsilon \sqrt{V(t, u)} \sum_{t_1 \leq s \leq t_2} \delta_n \frac{1}{2} (\Delta W(t_n, s)^2 - \Delta t_n), \\
    V(t_{n+1}, u) &= V(t_n, u) + [\kappa(\theta - V(t_n, u)) - \epsilon \sqrt{V(t_n, u)} \Upsilon^k_T(t_n, u)] \Delta t_n - \epsilon \sqrt{V(t_n, u)} \\
    &\quad - K \epsilon \sqrt{V(t_n, u)} \frac{1}{2} (\Delta W(t_n)^2 - \Delta t_n).
\end{align*}
$$

Notice that there is one more term for $L_k(t)$ which contains $V(t, u)$ since $V(t, u)$ is in the coefficients of $L_k(t)$. The caps/floors and swaptions prices can be calculated once $L_k(t)$s are simulated by Monte Carlo methods.

In this chapter, we have derived the multi-curve random fields volatility models. We first derived the dynamics of LIBOR forward rate $L_k(t)$ for random fields lognormal mixture model in Eq.(3.7), and the implied volatility formula in Eq.(3.12),
which can be used to capture the caplet volatility smile on the calibration procedure. Then we have also derived the dynamics of LIBOR forward rate $L_k(t)$ for random fields stochastic volatility models, in Eq.(3.16) for SABR type and in Eq.(3.19) for Wu-Zhang type respectively. Although there are no closed-form option pricing formulas for random fields stochastic volatility models, we can use the stochastic Taylor expansion to derive the discrete versions of dynamics, which can be used to price options numerically.
CHAPTER 4
MULTI-CURVE RANDOM FIELDS LIBOR MARKET MODEL

In this chapter we derive the multi-curve random fields LIBOR market model (MRFLMM), as well as the multi-curve random fields volatility models in the multi-curve framework. First, we show the inconsistency of similar interest rates after 2008 credit crunch in Sec.4.1 and review the multi-curve pricing methodology in Sec.4.2. Second, we derive the multi-curve random fields LIBOR market model in Sec.4.3, as well as the closed-form formulas for pricing European caplets and swaptions. Third, random fields volatility models in multi-curve framework are derived in Sec.4.5 and Sec.4.6, for local volatility models and stochastic volatility models respectively.

4.1 Interest Rate Modeling After Credit Crunch

After the credit crunch beginning in the summer of year 2007, the interest rates quoted in the market showed non-negligible inconsistency, which may exhibit arbitrage violation. Before the credit crunch, for example, the divergence between LIBOR based deposit rates and OIS rates for the same maturity chased each other with a safety distance of a few basis points. Similarly FRA rates and the corresponding forward rates implied by consecutive deposit rates would be quoted at a negligible spread. However, the pressure of a liquidity crisis in 2007 widened these spreads. Figure 4.1 shows the historical series of the U.S. LIBOR Deposit 6-month (6M) rates versus the U.S. Overnight Indexed Swap (OIS) 6-month (6M) rates over the time interval Mar.15.2005-Sep.14.2012 and Figure 4.2 reports the historical series of quoted U.S. Forward Rate Agreement (FRA) 3×6 rates versus the forward rates implied by the corresponding U.S. OIS 3M and 6M rates. We can see that the basis was well below ten basis points until summer 2007, but since then started moving erratically around different levels.

Figure 4.2. U.S. LIBOR FRA-3×6M Rates vs U.S. OIS-3×6M Fwd.Rates vs. Quotations Mar.15.2005-Sep.14.2012(source: Bloomberg)
The financial community has started the development of a new theoretical framework to deal with this problem. Morini [56] designed a theoretical framework that explained the divergence of such rates by introducing a stochastic default probability. However practitioners seemed to agree on an empirical approach, which was based on the construction of as many curves as possible tenors (e.g. 1-month, 3-month, 6-month). Future cash flows are thus generated through the curves associated to the underlying rates and then discounted by other curves. We can denote this approach as multi-curve methods. Mercurio [54], Kijima et al. [50], Chibane et al. [17], Henrard [32], Ametrano and Bianchetti [3], Ametrano [2] and Fujii et al. [21, 22, 23] have done many pioneer works using this approach.

4.2 Multi-curve Pricing Methodology

Before we introduce the multi-curve pricing methodology, we first review the traditional single curve pricing approach. First, we select one set of suitable interest instruments in market and build a single yield curve using the preferred bootstrapping procedure. Second, we compute the forward rates, cash flows, discount factors on this curve, under an appropriate measure. Third, we price the derivatives by summing up the discounted cash flows and hedge the resulting delta risk. However, this approach is not consistent with market practice anymore after the credit crisis. As pointed out in Bianchetti [9], it does not take into account the fact that the interest rate market is segmented into sub-areas corresponding to instruments with distinct underlying rate tenors. This fact may explain the interest rates inconsistency observed in market. For example, the divergence of LIBOR deposit rates and OIS rates, the spreads of FRA and the corresponding forward rates implied by consecutive deposits, and the basis swap spreads.

In practice the multiple curve approach mentioned in Sec.4.1 prevailed on the market. This approach is based on the construction of two different kinds of yield
curves, the discounting curve and forwarding curves. The single discounting curve is used to calculate discount factors and thus cash flow’s present values. The multiple forwarding curves, built from market instruments corresponding to the underlying tenor, are used to calculate future cash flows based on forward rates with the corresponding rate tenor. With this approach, interest rate derivatives with a given rate tenor should be priced and hedged using the instruments with the same underlying rate tenor. Ametrano and Bianchetti [3] and Bianchetti [9] summarized the multiple curve pricing methodology after credit crisis in the following procedure.

- Build one discounting curve $\zeta_d$ using the preferred choice of interest rate instruments and bootstrapping procedure.
- Build multiple forwarding curves $\zeta_{f_1}, \ldots, \zeta_{f_n}$ using preferred choice of distinct sets of interest rate instruments and bootstrapping procedure, with the same underlying LIBOR rate tenor for the corresponding curves.
- Compute the forward rates $F_{f_i}(t)$ with tenor $\tau_i$ using the corresponding forwarding curve $\zeta_{f_i}$.
- Compute the relevant discount factor $P(t, T)$ from the discounting curve $\zeta_d$.
- Compute the price of the derivative at time $t$ as the sum of the discounted cash flow $\sum_{k=1}^{m} P(t, T_k) E_{T_k} [c(F_{f_i}(t))]$, where $c(F_{f_i}(T))$ is the payoffs and the expectation is taken with respect to $T_k$-forward measure, associated with discount numeraire $P(t, T_k)$.

We use this methodology for our multiple curve pricing in this thesis. We can construct $i$ forwarding curves if the derivatives depend on the forward rates with $i$ different tenors. In this case we can assume that there are two kinds of curves, curve $\zeta_d$ for discounting with associated discount factor $P(t, T)$ and curves $\zeta_{f_i}$ for
future cash flows generating. The forward rates can be defined for the two different kinds of curves as in Eq.(2.16) with associated discount factors. The construction of the forward curve $\zeta_f$ is similar as in the pre-credit-crunch situation except that only the market quoted instruments corresponding to the tensor are employed in the bootstrapping procedure. For example, the three-month (3M) forward curve can be constructed by zero coupon rates from the 3M deposit rates, the 3M FRA and the 3M swap rates. The discount curve can be selected differently depending on contract to be priced. For example, it can be OIS curve if there is no counterparty risk or deposit rates if there exists risk. More details about the construction of yield curves will be discussed in Chapter 5 and Chapter 6.

Mercurio [54] extended the LIBOR market model (LMM) consistently with the two-curve assumption, namely joint evolutions of FRA rates and LIBOR rates. In the single curve case, a FRA rate can be defined as the expectation of the corresponding LIBOR rate under a given forward measure. However, in Mercurio’s multi-curve setting LIBOR rate and the forward measure belong to different curves. FRA rates are thus different from LIBOR rates and can be modeled with their own dynamics.

By Mercurio [54], to show how to value the interest rate derivatives in two-curve setting of distinct forwarding and discounting curves, we consider an interest rate swap where the floating leg pays the LIBOR forward rates $L^f_k(t)$ from forward curve $\zeta_f$. The time $t$ value of the floating leg payoff is

$$\delta_k P(T, T_k)E^{T_k}[L^f_k(T_{k-1})|\mathcal{F}_t],$$

(4.1)

where the expectation is under the forward measure associated with discount curve $\zeta_d$. Define the time $t$ forward rate agreement (FRA) rate $F^f_k(t)$ as the fixed rate to be exchanged at time $T_k$ for floating payment $\delta_k L^f_k(T_{k-1})$ so that the swap rate has zero value at time $t$, i.e.

$$F^f_k(t) := E^{T_k}[L^f_k(T_{k-1})|\mathcal{F}_t].$$

(4.2)
The net present value of the swap’s floating leg is given by summing the single payoffs

$$\sum_{k=i+1}^{j} \delta_k P(t, T_k) F_{k}^{I_j}(t).$$  \hspace{1cm} (4.3)

Thus the swap rate is

$$S_{i,j}(t) = \frac{\sum_{k=i+1}^{j} \delta_k P(t, T_k) F_{k}^{I_j}(t)}{\sum_{k=i+1}^{j} \delta_k P(t, T_k)}.$$  \hspace{1cm} (4.4)

We can see that the formula of swap rate, Eq.(4.4), in the multi-curve framework, is different from the one in the single-curve framework, Eq.(1.25). Indeed, the multi-curve framework reduces to the single-curve framework when the discounting curve and forwarding curves coincide, i.e., $\zeta_f = \zeta_d$, for all $i$. In the single-curve framework, $L_k(t)$ is a martingale under the associated $T_k$-forward measure $Q^{T_k}$, thus we have that $F_{k}^{I_j}(t) = L_{k}^{I_j}(t)$, which makes Eq.(4.3) to become $P(t, T_i) - P(t, T_j)$. Hence Eq.(4.4) becomes

$$S_{i,j}(t) = \frac{P(t, T_i) - P(t, T_j)}{\sum_{k=i+1}^{j} \delta_k P(T, T_k)},$$  \hspace{1cm} (4.5)

which is exactly the formula of swap rate in single-curve framework.

In this chapter we extend the random fields LIBOR market model to the multi-curve framework. For simplicity we investigate the case that there is only one curve used to generate future cash flows and the curve used for discounting is different from it. We follow the approach of Brace et al. [12] and Miltersen et al. [55] by assuming that forward LIBOR rates have lognormal diffusions and then extend the random field LIBOR market model to multi-curve framework in the spirit of Mercurio [54].

### 4.3 Multi-curve Random Fields LIBOR Market Model

In this section we derive the dynamics of LIBOR rates $L_k(t)$ from discount curve and FRA rates $F_{k}^{I_j}(t)$ from forward curve with uncertainty term modeled with random fields under risk neutral measure Q and $T_j$-forward measure $Q^{T_j}$ for $j = 1, ..., N$. 
Let us consider time structures \( \{ T_0, T_1, \ldots, T_N \} \) with intervals \( \delta_k = T_k - T_{k-1}, \)
\( 1 \leq k \leq N \). For \( t < T_{k-1} < T_k \), in two-curve setting we need to model the evolution of FRA rates \( F^f_k(t) \) from forward curve \( \zeta_f \) under \( T_k \)-forward measure:
\[
dF^f_k(t) = F^f_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u) dB^{T_k}(t, u) du, \tag{4.6}
\]
where \( B^{T_k}(t, u) \) is a Brownian field with correlation
\[
corr[dB(t, T_1), dB(t, T_2)] = c_f(T_1, T_2).
\]
Suppose that the LIBOR forward rates \( L_k(t) \) from discount curve defined in Eq.(1.12) has dynamics
\[
dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_k}(t, u) du, \tag{4.7}
\]
where
\[
corr[dW(t, T_1), dW(t, T_2)] = c_d(T_1, T_2).
\]
In addition we also need to specify the correlation between the two random fields
\[
corr[dB(t, T_1), dW(t, T_2)] = c_{df}(T_1, T_2).
\]
By Brigo and Mercurio [13], for \( k < j \), in two-curve case, the FRA rate \( F^f_k(t) \) has the drift term
\[
-dF^f_j(t) d \ln \frac{P(t, T_j)}{P(t, T_k)} = -dF^f_j(t) \ln(1/[ \prod_{i=k+1}^{j} (1 + \delta_i L_i(t))])
\]
\[
= \sum_{i=k+1}^{j} \delta_i \frac{dF^f_j(t) dL_i(t)}{dt}
\]
\[
= F^f_j(t) \int_{T_{j-1}}^{T_j} \eta_j(t, u)[ \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, u) c_{df}(v, u)}{1 + \delta_i L_i(t)} dv] du.
\]
The derivation in case \( j < k \) is analogous. Thus we have the following theorem.

**Theorem 4.3.1.** (Two-curve random fields dynamics under forward measures) Under the lognormal assumptions, the dynamics of the instantaneous forward rates \( L_k(t) \) and FRA rates \( F^f_k(t) \) under the \( T_j \)-forward measure, in three cases
\( j < k, j = k, j > k, \) are described respectively by

\[
\begin{align*}
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^T_j(t,u) + \Lambda^j_k(t,u)dt]du, \\
    dF^j_k(t) &= F^j_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t,u)[dB^T_j(t,u) + \Lambda^j_k(t,u)dt]du, \quad j < k; \\
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)dW^T_j(t,u)du, \\
    dF^j_k(t) &= F^j_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t,u)dB^T_j(t,u)du, \quad j = k; \\
    dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^T_j(t,u) + \Lambda^j_k(t,u)dt]du, \\
    dF^j_k(t) &= F^j_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t,u)[dB^T_j(t,u) + \Lambda^j_k(t,u)dt]du, \quad j > k;
\end{align*}
\]

(4.8)

with

\[
\Lambda^j_k(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t,v) c_{df}(v,u)}{\delta_i L_i(t) + 1} dv,
\]

and

\[
\Lambda^j_k(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t,v) c_{df}(v,u)}{\delta_i L_i(t) + 1} dv,
\]

where \( W^T_j(t,u) \) is a Gaussian random field under \( T_j \)-forward measure. The above equations admit a unique solution if the coefficient \( \xi_k(\cdot, \cdot) \) are locally bounded, locally Lipschitz continuous and predictable.

4.4 Option Pricing in Multi-curve Random Fields LIBOR Market Model

In this section we derive the closed-form Black implied volatility formulas for caplets and swaptions in multi-curve random field LIBOR market model, using two-curve setting as an example.

4.4.1 Multi-curve random fields LMM formula for caplets. The payoff of a caplet at time \( T_k \) is

\[
\delta_k[L^k_k(T_{k-1}) - K]^+.
\]
The caplet price at time $t$ becomes

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E^{T_k} [[L_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t].$$

Since in two curve setting the pricing measure is the $T_k$-forward measure for discount curve, the LIBOR forward rate $L_k^f(t)$ from forward curve is not a martingale under the measure $\mathbb{Q}^{T_k}$. Mercurio [54] introduced a way to solve it. From the definition of FRA rate Eq.(4.2), we know that

$$F_k^f(T_{k-1}) = L_k^f(T_{k-1}),$$

which means that the price for caplet in two curve setting can be written as

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E^{T_k} [[F_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t]$$

Suppose that the FRA rate $F_k^f(t)$ has dynamics Eq.(4.8) under $T_k$-forward measure, then the price is given as

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \text{Black}(K, F_k^f(t), \sigma_{k,MR}^\text{Black} \sqrt{T_{k-1} - t}),$$

where

$$\sigma_{k,MR}^\text{Black} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} \eta_k(s, x) \eta_k(s, y) c_f(x, y) dx dy ds.} \quad (4.9)$$

4.4.2 Multi-curve random fields LMM formula for swaptions. The payoff of a swaption at time $T_k$ is

$$\delta_k [S_{i,j}(T_{k-1}) - K]^+. $$

The time $t$-price is therefore

$$[S_{i,j}(t) - K]^+ \sum_{k=i+1}^j \delta_k P(t, T_k),$$
where

\[
S_{i,j}(t) = \frac{\sum_{k=i+1}^{j} \delta_k P(t, T_k) F_k^j(t)}{\sum_{k=i+1}^{j} \delta_k P(t, T_i)} = \sum_{k=i+1}^{j} \frac{P(t, T_k)/P(t, T_i)}{\delta_k P(t, T_k)/P(t, T_i)} \delta_k F_k^j(t) = \sum_{k=i+1}^{j} \frac{\delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_i(t))}{\delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_j(t))} F_k^j(t)
\]

is the swap rate. Analogous to the discussion in Sec.2.4.2, we have

\[
\ln S_{i,j}(t) = \ln \left[ \sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_i(t)) F_k^j(t) \right] - \ln \left[ \sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j} (1 + \delta_l L_j(t)) \right].
\]

By Itô’s formula, we know that

\[
\frac{1}{S_{i,j}(t)} \frac{\partial S_{i,j}(t)}{\partial L_i(t)} = \frac{\alpha_{i,j}^j(t) \delta_k}{1 + \delta_k L_i(t)} S_{i,j}(t) \frac{\partial S_{i,j}(t)}{\partial F_k^j(t)} = \frac{\beta_{i,j}^j(t) \delta_k}{1 + \delta_k L_i(t)},
\]

where

\[
\alpha_{i,j}^j(t) = \frac{\sum_{h=i+1}^{k-1} \delta_h F_h^j(t) \prod_{l=h+1}^{j} (1 + \delta_l L_i(t))}{\sum_{h=i+1}^{j} \delta_h F_h^j(t) \prod_{l=h+1}^{j} (1 + \delta_l L_i(t))} - \frac{\sum_{h=i+1}^{k-1} \delta_h \prod_{l=h+1}^{j} (1 + \delta_l L_i(t))}{\sum_{h=i+1}^{k-1} \delta_h \prod_{l=h+1}^{j} (1 + \delta_l L_i(t))},
\]

\[
\beta_{i,j}^j(t) = \frac{\prod_{h=k}^{j} (1 + \delta_h L_i(t))}{\sum_{h=i+1}^{j} \delta_h F_h^j(t) \prod_{l=h+1}^{j} (1 + \delta_l L_i(t))}.
\]

Using Itô’s formula for two variables, the uncertainty term of \(dS_{i,j}(t)\) is given by

\[
\sum_{k=i+1}^{j} \frac{1}{S_{i,j}(t)} \frac{\partial S_{i,j}(t)}{\partial L_k} \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_k}(t, u) du + \frac{\partial S_{i,j}(t)}{\partial F_k^j} \int_{T_{k-1}}^{T_k} \eta_k(t, u) dB^{T_k}(t, u) du,
\]

or equivalently

\[
\sum_{k=i+1}^{j} \int_{T_{k-1}}^{T_k} \frac{\alpha_{i,j}^j(t) \delta_k L_k(t) \xi_k(t, u)}{1 + \delta_k L_k(t)} dW^{T_k}(t, u) + \frac{\beta_{i,j}^j(t) \delta_k L_k(t) \eta_k(t, u)}{1 + \delta_k L_k(t)} dB^{T_k}(t, u) du.
\]
Thus the Black implied volatility of $S_{i,j}(t)$ is defined as

\[
\sigma_{i,j}^{\text{Black,MR}}(t) = \sqrt{\frac{1}{T_t - t} \int_t^{T_t} \left\| \sum_{k=1+1}^j \int_{T_k-1}^{T_k} \left[ \alpha_k^{i,j}(s) \frac{\delta_k L_k(s) \xi_k(s, u)}{1 + \delta_k L_k(s)} \right] dW^T_k(s, u) \right\|^2 ds + \beta_k^{i,j}(s) \eta_k(s, u) dB^T_k(s, u) ds} \]

\[
= \sqrt{\frac{1}{T_t - t} \int_t^{T_t} \left\| \sum_{k=1+1}^j \int_{T_k-1}^{T_k} \left[ \alpha_k^{i,j}(s) \xi_k(s, u) dW^T_k(s, u) \right] \right\|^2 ds + \beta_k^{i,j}(t) \eta(s, u) dB^T_k(s, u) ds} \]

\[
= \sqrt{\frac{1}{T_t - t} \sum_{k=1+1}^j \sum_{l=1+1}^j \int_{T_k-1}^{T_k} \left\{ \int_t^{T_k} \left[ \alpha_k^{i,j}(s) \xi_k(s, u) \right] dW^T_k(s, u) \right\} ds + \beta_k^{i,j}(t) \eta(s, u) dB^T_k(s, u) ds} \]

\[
\xi_k(s, u) \alpha_k^{i,j}(s) \xi_l(s, v) c_d(u, v) + \beta_k^{i,j}(t) \eta_k(s, u) \beta_k^{j,l}(s) \eta_l(s, v) c_f(u, v)
\]

\[
+ \alpha_k^{i,j}(s) \xi_k(s, u) \alpha_k^{l,j}(s) \xi_l(s, v) c_{df}(u, v)
\]

\[
+ \beta_k^{i,j}(s) \eta_k(s, u) \alpha_k^{l,j}(s) \xi_l(s, v) c_{df}(u, v) dudv \right\} ds. \tag{4.10}
\]

The third equation is obtained by using standard freezing approximation techniques, i.e., approximately evaluating the LIBOR rates $L_k(s)$, $t \leq s \leq T_t$, appearing in the instantaneous volatility at initial time $t$.

The above equation is too complex. To simplify the formula we resort to a simple approximation technique. We know that the swap rate $S_{i,j}(t)$ can be written as a linear combination of FRA rates $F^f_k(t)$:

\[
S_{i,j}(t) = \sum_{k=1+1}^j \alpha_k^{i,j}(t) F^f_k(t),
\]
with
\[ \omega_k^{i,j}(t) = \frac{\delta_k P(t, T_k)}{\sum_{k=i+1}^j \delta_k P(t, T_i)}. \]

We freeze the weights \( \omega_k^{i,j}(t) \) at their time \( t \) value and denote it by \( \omega_k^{i,j} \). Thus we have the approximation:
\[ S_{i,j}(t) = \sum_{k=i+1}^j \omega_k^{i,j} F_k^f(t), \]
which leads to
\[ dS_{i,j}(t) = \sum_{k=i+1}^j \omega_k \int_{T_{k-1}}^{T_k} \eta_k(t, u) F_k^f(t) dB_k^T(t, u) du. \]

Notice that by freezing the weights we are in fact freezing the dependence of \( S_{i,j}(t) \) on \( L_k(t) \). Thus the Black implied volatility is defined approximately as:
\[ \sigma_{i,j}^{Black, MR} = \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \left\| \sum_{k=i+1}^j F_k^f(s) \omega_k^{i,j} \int_{T_{k-1}}^{T_k} \eta_k(s, u) dB(s, u) \right\|^2 ds} \]
\[ = \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j F_k^f(t) \omega_k^{i,j} F_l^f(t) \omega_l^{i,j}} \]
\[ \times \int_t^{T_i} \int_{T_{k-1}}^{T_k} \int_{T_{l-1}}^{T_l} \eta_k(t, x) \eta_l(t, y) c_f(x, y) dx dy ds. \] \[ (4.11) \]

### 4.5 Multi-curve Random Fields Local Volatility Models

In this section we extend the random field lognormal mixture model derived in Sec.3.2 to the multi-curve framework. Analogous to Eq.(4.6), we can assume that under \( T_k \)-forward measure, the dynamics of \( L_k(t) \) and \( F_k^f(t) \) with random field are given as
\[ \begin{cases} dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW_k^T(t, u) du; \\ dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, F_k^f(t), u) dB_k^T(t, u) du. \end{cases} \]
Analogous to the discussion in Sec.3.2, we can derive a random field lognormal mixture model for $F^f_k(t)$. Let us consider the diffusion process with dynamics given by

$$
\frac{dH^k(t)}{H^k(t)} = \int_{T^k_{k-1}} \mu^k_i(t, H^k(t), u) dB^k(t, u) du
$$

with initial value $H^k_i(0) = F^f_i(0)$ for all $i = 1, 2, ..., M$, where $B^k(t, u)$ is a Gaussian field under $T^k$-forward measure with correlation $c_f(T_1, T_2)$ Analogous to Brigo and Mercurio [13], the problem here is to derive the local volatility $\eta_k(t, u)$ such that the density of $F^f_k(t)$ under $T^k$-forward measure satisfies, for each time $t$:

$$
q^k_i(x, t) = \frac{d}{dx} P^T_k \{ F^f_k(t) \leq x \} = \sum_{i=1}^M \omega_i \frac{d}{dx} P^T_k \{ H^k_i(t) \leq x \} du = \sum_{i=1}^M \omega_i q^k_i(x, t),
$$

where $\omega_i$ is a weight function with $\sum_{i=1}^M \omega_i = 1$. In fact $q^k_i(x)$ is a proper density function under $T^k$-forward measure since

$$
\int_0^{+\infty} x q^k_i(x, t) dx = \int_0^{+\infty} x \sum_{i=1}^M \omega_i q^k_i(x, t) dx = \sum_{i=1}^M \omega_i H^k_i(0) = F^f_k(0).
$$

The last calculation comes from the fact that $H^k_i(t)$ is a martingale under $T^k$-forward measure. We know that the local volatility $\eta_k(t, F^f_k(t))$ is

$$
\int_{T^k_{k-1}}^{T_k} \eta_k(t, u) \eta_k(t, v) c_f(t, u, v) du dv
$$

$$
= \sum_{i=1}^M \omega_i \left[ \int_{T^k_{k-1}}^{T_k} \mu^k_i(t, u) \mu^k_i(t, v) c_f(t, u, v) du dv \right] q^k_i(x, t)
$$

$$
= \sum_{i=1}^M \omega_i q^k_i(x, t) \left[ \int_{T^k_{k-1}}^{T_k} \mu^k_i(t, u) \mu^k_i(t, v) c_f(t, u, v) du dv \right].
$$

If we take $c_f(t, x, y) = 1$, the above formula reduces to

$$
\eta_k(t) = \sqrt{\frac{\sum_{i=1}^M \omega_i \mu^k_i(t) q^k_i(x, t)}{\sum_{i=1}^M \omega_i q^k_i(x, t),}}
$$

which is the original lognormal-mixture model derived in Brigo and Mercurio [13].

Since

$$
\int_{T^k_{k-1}}^{T_k} \eta_k(t, F_k(t), u) dB^k(t, u) du
$$
has normal distribution with variance
\[ \int_{T_{k-1}}^{T_k} \eta_k(t, F_k(t), u) \eta_k(t, F_k(t), x) c_d(t, u, v) dv du, \]
we have following theorem:

**Theorem 4.5.1. (Random fields lognormal-mixture model dynamics under forward measures)** The dynamics of FRA rate \( F^f_k(t) \) is given by

\[
dF^f_k(t) = F^f_k(t) \left( \sum_{i=1}^{M} \omega_i \left( \int_{T_{k-1}}^{T_k} \mu^k_i(t, u) \mu^k_i(t, v) c_f(t, u, v) dv du \right) q^{k_i}(x, t) \right) dB^{T_k}(t),
\]

where \( W^{T_k}(t) \) is a Brownian motion under \( T_k \)-forward measure.

If we assume that in Eq. (4.12)

\[
\mu^k_i(t, x, u) = \mu^k_i(t, u),
\]

i.e. the densities \( p^k_i(x, t) \) are all lognormal, where for all \( k \), \( \mu^k_i(t) \) are deterministic and continuous functions of time that are bounded from above and below by strictly positive constants, then the marginal density of \( G^k_i(t) \) is lognormal and given by

\[
q^{k_i}(x, t) = \frac{1}{x U^k_i(t) \sqrt{2\pi}} \exp\left\{ -\frac{1}{2 U^k_i(t)^2} \left[ \ln \frac{x}{F_k(0)} + \frac{1}{2} F^k_i(t)^2 \right] \right\};
\]

\[
U^k_i(t) = \sqrt{\int_0^t \int_{T_{k-1}}^{T_k} \mu^k_i(s, u) \mu^k_i(s, v) c_d(s, u, v) dv du ds}.
\]

**4.5.1 Option pricing in random fields local volatility models.** In this section, we will derive the closed-form pricing formulas of caplets in two-curve random field lognormal mixture model.

The payoff of a caplet at time \( T_k \) is

\[
\delta_k [L_k(T_{k-1}) - K]^+.
\]
Following the discussion in Sec.4.4.1, the caplet price at time $t$ becomes

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E^{T_k}[[L_k(T_{k-1}) - K]^+ | \mathcal{F}_t]$$

$$= \delta_k P(t, T_k) E^{T_k}[[F_k(T_{k-1}) - K]^+ | \mathcal{F}_t]$$

$$= P(0, T_k) \int_0^{+\infty} [x - K]^+ q_k(x, t) dx$$

$$= P(0, T_k) \sum_{i=1}^M \int_0^{+\infty} [x - K]^+ q^k_i(x, t) dx.$$

(4.18)

Suppose that the FRA rate $F^l_k(t)$ has dynamics as Eq.(4.12) under $T_k$-forward measure. Then the caplet price is given as

$$C_{\text{plt}}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \sum_{i=1}^M \text{Black}(K, F_k(t), U^k_i(T_{k-1}, u)).$$

Given the above analytical tractability, we can derive an explicit approximation for the caplet implied volatility as a function of the caplet strike.

**Theorem 4.5.2. (Implied volatility of random field lognormal-mixture model)** Define $m = \ln \frac{F_k(t)}{K}$. The implied volatility $\sigma_{k,\text{MRF}}^{\text{Black}}$ is

$$\sigma_{k,\text{MRF}}^{\text{Black}}(m) = \sigma_{k,\text{MRF}}^{\text{Black}}(0) + \frac{1}{2\sigma_{k,\text{MRF}}^{\text{Black}}(0)(T_{k-1} - t)} \sum_{i=1}^M \omega_i \left[ \sigma_{k,\text{MRF}}^{\text{Black}}(0) \sqrt{T_{k-1} - t} V^k_i(T_{k-1}) \right]$$

$$e^{\frac{1}{2}(\sigma_{k,\text{MRF}}^{\text{Black}}(0))^2(T_{k-1} - t) - U^k_i(T_{k-1})^2 - 1]m^2 + o(m^2),$$

where the at-the-money implied volatility $\sigma_{k,\text{MRF}}^{\text{Black}}(0)$ is given by

$$\sigma_{k,\text{MRF}}^{\text{Black}}(0) = \frac{2}{\sqrt{T_{k-1} - t}} \Phi^{-1} \left( \sum_{i=1}^M \omega_i \Phi \left( \frac{1}{2} U^k_i(T_{k-1}) \right) \right).$$

(4.19)

4.6 Multi-curve Random Fields Stochastic Volatility Models

In this section we derive the approximate solution for caplets and swaptions in the case of stochastic volatility. We take the stochastic volatility structures as an example, one by Hagan et al.[27] (SABR) and one by Wu and Zhang [70].
4.6.1 SABR approach. Analogous to Sec.3.3.1, it is natural to model the FRA rate in two curves case as

\[
\begin{align*}
\frac{dF_{f}^j}{f}(t) &= F_{f}^j(t)\beta_{k}' \int_{T_{k-1}}^{T_{k}} V_{k}'(t, u)dB_{T_k}(t, u)du, \\
\frac{dV'}{u}(t, u) &= \alpha_{k}'(t)V_{k}'(t, u)dB_{T_k}(t, u),
\end{align*}
\]

with correlation structure

\[
\text{corr}[dB(t, T_1), dB(t, T_2)] = c_f(T_1, T_2),
\]

where \(\beta_k' > 1, \alpha'\) are positive constants. Analogous to previous works, the dynamics of \(F_{f}^j(t)\) and \(V_{k}'(t, u)\) under the \(T_j\)-forward measure, in three cases \(j < k, j = k, j > k\), are described respectively by

\[
\begin{align*}
\frac{dF_{f}^j}{f}(t) &= F_{f}^j(t)\beta_{k}' \int_{T_{k-1}}^{T_{k}} V_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt, \\
\frac{dV'}{u}(t, u) &= V_{k}'(t, u)\alpha_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt, \\
\frac{dF_{f}^j}{f}(t) &= F_{f}^j(t)\beta_{k}' \int_{T_{k-1}}^{T_{k}} V_{k}'(t, u)dB_{T_k}(t, u), \\
\frac{dV}{k}(t, u) &= V_{k}(t, u)\alpha_{k}'(t, u)dB_{T_k}(t, u), \\
\frac{dF_{f}^j}{f}(t) &= F_{f}^j(t)\beta_{k}' \int_{T_{k-1}}^{T_{k}} V_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt, \\
\frac{dV_{k}'}{u}(t, u) &= V_{k}'(t, u)\alpha_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt, \\
\frac{dF_{f}^j}{f}(t) &= F_{f}^j(t)\beta_{k}' \int_{T_{k-1}}^{T_{k}} V_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt, \\
\frac{dV_{k}'}{u}(t, u) &= V_{k}'(t, u)\alpha_{k}'(t, u)dB_{T_k}(t, u) + \Psi_{j}^{k}(t, u)dt,
\end{align*}
\]

with

\[
\Psi_{j}^{k}(t, u) = \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_{i}} \frac{\delta_i L_i(t)\beta_{k}' V_{k}(t, v)c_{df}(t, v, u)}{\delta_i L_i(t) + 1} dv.
\]

As in Eq.(3.16), the model above does not have analytic closed-form solutions and we can present a version of stochastic Taylor expansion in the spirit of Milstein scheme in random field setting. The formulas are similar to those derived in Sec.3.3.1.

4.6.2 Wu-Zhang approach. Analogous to Sec.3.3.2, it is natural to model the
FRA rate in two curves case as

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dF^I_k(t)}{F^I_k(t)} = \sigma_k F^I_k(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} dB^T_k(t, u) du, \\
\frac{dV'(t, u)}{du} = \kappa'(\theta' - V'(t, u))dt + \epsilon' \sqrt{V'(t, u)} dB^T(t, u),
\end{array} \right. \\
\end{align*}
\]

with

\[
\text{corr}(dB(t, u), dB(t, v)) = c(t, u, v),
\]

where \( \sigma_k', \kappa', \theta', \epsilon' \) are positive constants. Since the measure change does not change the correlation of Brownian motion, the above correlation can exist under both measures, \( Q^T_k \) and \( Q^T \). The dynamics of \( F^I_k(t) \) and \( V^I_k(t, u) \) under the \( T_j \)-forward measure, in three cases \( j < k, j = k, j > k \), are described respectively by

\[
\begin{align*}
\frac{dF^I_k(t)}{F^I_k(t)} &= \sigma_k F^I_k(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} dB^T_j(t, u) du + \Upsilon^j_{k+1}(t, u) dt, \\
\frac{dV'(t, u)}{du} &= \kappa'(\theta' - V'(t, u))dt - \epsilon' \sqrt{V'(t, u)} dB^T_j(t, u) + \Upsilon^j_{k+1}(t, u) dt, \\
\frac{dF^I_k(t)}{F^I_k(t)} &= \sigma_k F^I_k(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} dB^T_k(t, u) du, \\
\frac{dV'(t, u)}{du} &= \kappa'(\theta' - V'(t, u))dt - \epsilon' \sqrt{V'(t, u)} dB^T_k(t, u) + \Upsilon^j_{k+1}(t, u) dt, \\
\end{align*}
\]

with

\[
\Upsilon^j_{k+1}(t, u) = \sum_{i=1}^{j} \int_{T_{i-1}}^{T_i} \delta_i \sigma_i L_i(t) \sqrt{V(t, u)} c_{dq}(t, v, u) \frac{1 + \delta_i L_i(t)}{dv}.
\]

As in Eq.(3.19), the model above does not have analytic closed-form solutions and we can present a version of stochastic Taylor expansion in the spirit of Milstein scheme in random field setting. The formulas are similar to those derived in Sec.3.3.2.

In this chapter, we have derived the random fields volatility models. We first derived the dynamics of LIBOR forward rate \( L_k(t) \) for random fields lognormal mixture model in Eq.(4.8), and the implied volatility formula in Eq.(4.14), which can
be used to capture the caplet volatility smile on the calibration procedure. Then we derived the dynamics of LIBOR forward rate $L_k(t)$ for multi-curve random fields stochastic volatility models, in Eq.(4.21) for SABR type and in Eq.(4.21) for Wu-Zhang type respectively. Although there are no closed-form option pricing formulas for random fields stochastic volatility models, we can use the stochastic Taylor expansion to derive the discrete versions of dynamics, which can be used to price options numerically.
CHAPTER 5
CALIBRATION

In this chapter we calibrate four models, single-curve LMM, single-curve R-FLMM, multi-curve LMM and multi-curve RFLMM to market quoted caps and swaptions prices on selected trading days. First, in Sec.5.1 we review the calibration procedure of interest rate models based on forward LIBOR rates. Second, we compare the calibration results of the four models in Sec.5.2, including the pricing and hedging performance.

5.1 Calibration of LIBOR Interest Rate Models

The calibration of interest rate models is the computation of the parameters of the models, so as to match as closely as possible model derived prices to market observed/quoted prices of actively traded instruments. During the calibration process, one may use statistical or computational methods. Usually there are three different sources of information which can be used for the model calibration. These sources are current term structure, market quoted caps or swaptions volatilities and historical forward rate correlation matrix.

The market models we discuss and derive in this thesis require in general three different inputs, the initial curve and forward curve, the instantaneous volatility, and the correlation structure. The ways to get the inputs are discussed as follows. 1)The initial yield curve and the corresponding forward rates can be bootstrapped from market zero-coupon bond prices. However, in multi-curve framework, the discount curve is distinct from forward curve and there are different forward curves needed for different tenors. Thus the bootstrapping of discount curve and forward curves should be investigated carefully. See Sec.5.1.1 for details. 2)The instantaneous volatilities of forward rates are usually assumed to depend on current time $t$, forward rate maturity
The goal of LIBOR interest rate models calibration is to estimate the instantaneous volatility functions $\xi_k(t)$, $k = 1, 2, ..., N$ and the correlation matrix $\rho_{i,j}(t)$, $i, j = 1, 2, ..., N$ from the data of the initial yield curve $L_k(0)$ and the caps and swaptions prices observed on the market. Notice that in random fields case, the correlation structure may take the continuum form $c(u, v)$ and in multi-curve framework, we use different curves to bootstrap the discounting curve and the forwarding curves. In this thesis, we use parametric forms to describe the instantaneous volatility $\xi_k(t)$ and the correlation structure $c(u, v)$.

5.1.1 The construction of different curves. As discussed in Chapter 4, the spirit of multi-curve modeling is that interest rate derivatives should be priced using instruments with the same underlying rate tenor. In the HJM multi-curve framework, there are one discounting curve and many forwarding curves, one for each quoted LIBOR rate tenor. The problem of building different curves corresponding to different rate tenors has been addressed in the work of some pioneers. For example, Bianchetti
treats the different curves as if they are curves for different currencies. Pallavicini and Tarenghi bootstrap the yield curves by interpolating on the spreads between modified forward rates of different tenors, along with their spread with respect to forward calculated from discounting curve. In this thesis we follow the approach of Ametrano and Bianchetti, where they solve the problem in terms of different yield curves coherent with basic derivatives.

Since the swaptions in our dataset are based on swap rates on LIBOR-6M, we only construct one forwarding curve, which has tenor 6M. Following the spirit in Ametrano and Bianchetti and the discussion in Chapter 4, in multi-curve framework, the discounting curve is obtained from U.S. Overnight Indexed Swaps from one to thirty years and the forwarding curve is bootstrapped from LIBOR-6M fixing, FRA rates up to two years, swaps from two to thirty years paying an annual fixed rate in exchange for the LIBOR-6M rate, using the OIS curve as discounting curve. By Pallavicini and Tarenghi, the choice of discounting curve using OIS rates can be justified by the fact that interbank operations are usually collateralized and it is straightforward to use overnight rate for discounting if we assume that the collateral is revalued daily. On the other hand, in the single curve framework, the yield curve used for both discounting and forwarding are obtained as usual, short term LIBOR deposits(blew 1 year), mid-term FRA on LIBOR 3M (below 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). The details of strapping procedure can be found in Ametrano and Bianchetti and Bianchetti et al. We can build the curves as follows.

1. LIBOR standard curve: the classic yield curve bootstrapped from short term LIBOR deposits(blew 1 year), mid-term FRA on LIBOR 3M (below 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). In single-curve modeling this curve will be used for discounting and forwarding curve are bootstrapped
using as discounting curve.

2. OIS curve: the curve bootstrapped from the U.S. OIS rates. In two-curve modeling this curve will be used as discount curve.

3. LIBOR 6M curve: the LIBOR-OIS 6M curve bootstrapped from the LIBOR deposit 6M, mid-term FRA on LIBOR-6M(up to 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). In two curve modeling this curve will be used as forward curve.

5.1.2 The choice of the instantaneous volatility. The desired qualitative features of the instantaneous volatility usually come from the term structure of volatility which is directly observable from market. Volatilities generally depend on three factors: current time \( t \), maturity \( T \) and time to maturity \( \tau = T - t \). By Rebonato [64] the forward rate volatilities can also be deterministic functions of the full history of yield curve and its stochastic drivers or some stochastic quantities whose future values are known only in a statistical sense. Furthermore the instantaneous volatility could itself be a diffusion process. The most popular assumption is the time-homogeneity of the volatilities. Rebonato [63] proposed a time homogenous linear-exponential functional form with four parameters \( \xi_k(t) = f(t, T_k)h(t)g(T_k) \) for the instantaneous volatility, while \( h(t), g(T_k) \) are usually taken to be 1 and

\[
f(t, T_k) = [a + b(T_k - t)]e^{-c(T_k-t)} + d; a, b, c, d > 0. \tag{5.1}
\]

Thus \( \xi_k(t) = f(t, T_k) \). By Rebonato [63], this formulation is reasonable since it satisfies criteria of desired volatility. First, the function is flexible enough to be able to produce either a humped or monotonically decreasing instantaneous volatility. Second, the function parameters have clear econometric interpretation to allow sanity check of calibration. For example, the parameter \( d \) is the volatility with large number
of $\delta = T - t$, while the amount $a + d$ is approximately the instantaneous volatility of the forward rate with small $\delta$. And the extreme of the function is reached at $(b - ca)/cb$. Third, analytical integration of the functions square should be possible to allow fast calculation of forward rate variance and covariance.

5.1.3 The specification of correlation structure. The correlation structure differs for LMM and random fields LMM. The LMM needs the specification of the functional form of correlation matrix $\rho_{i,j}$, while the random field LMM needs the correlation functions $c(u, v)$. The difference of correlation structures for LMM and random fields LMM is discussed as follows.

For LMM, given the dynamics in Eq.(1.18) we know that the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\frac{\text{Cov}(dL_i(t), dL_j(t))}{\sqrt{\text{Var}(dL_i(t))\text{Var}(dL_j(t))}} = \frac{\xi_i(t)\xi_j(t)dW_i(t)dW_j(t)}{\sqrt{\xi_i^2(t)\xi_j^2(t)}} = dW_i(t)dW_j(t) = \rho_{i,j}(t),$$

which means that the instantaneous correlation of forward rates is exactly the same as the correlation structure of Brownian motion $W(t)$. Thus the correlation matrix $\rho(t)$ should have the desirable features of the forward rates correlation structures, which is the historical correlation matrix of forward rates quoted in market. Meanwhile, the historical correlation matrix has two main features. First, the correlation matrix should always be decreasing when moving away from diagonal, which corresponds to the fact that forward rates with maturities close to each other tend to be more likely to move to the same direction as the rates with distinct maturities. Second, the correlation matrix should be increasing along sub-diagonal. We know that yield curve tends to flatten and forward rates with long maturities generally move in a more correlated way than rates with short maturities. Thus the parametric correlation matrix should fulfill such two features.
The classical function form to describe correlation is

\[ \rho_{i,j} = \rho_0 + (1 - \rho_0)e^{-\rho_\infty|i-j|}, \]  

(5.2)

where \( \rho_0 \) and \( \rho_\infty \) are positive. However, the problem of classical form is that it is not increasing along the sub-diagonal. To solve this problem, many authors created different forms of parametric correlation matrix. For example, Schoenmakers and Coffey [66] proposed a two-parameter function form to describe the correlations:

\[ \rho_{i,j} = e^{-\frac{|i-j|}{N-1}(\rho_\infty + \rho_0 \frac{N-1-|j-i|}{N-2})}, \]  

(5.3)

where \( N \) is the size of the matrix \( \rho \) and \( 0 \leq \rho_0 \leq \rho_\infty \). The correlation structure Eq.(5.3) can capture the two features of the correlation of forward rates and we use Eq.(5.3) as our parametric form for correlation matrix in calibration. As shown in Sec.1.4, the number of factors in LIBOR market model depends on the rank of correlation matrix \( \rho \). A rank-\( d \) correlation matrix \( \rho \) with entries defined in Eq.(5.3) gives rise to a \( d \)-factor LIBOR market model. In this thesis we take a full rank correlation matrix and thus the LIBOR market model considered in this thesis has the number of factors the same as the number of forward rates considered. Notice that if we specify the correlation directly, we need to estimate about \( \frac{K(K+1)}{2} \) parameters in estimation procedure, where \( K \) is the size of the correlation matrix. The number of parameters will become very large if \( K \) is large. Thus the techniques of factor reduction will be apply to reduce the rank of the matrix and thus the number of parameters. In this thesis, we use functional form to describe the correlation structure. The parameters needed to be estimated are just the parameters from the functional form. Thus it is not necessary to reduce the correlation matrix to low-rank.

However, for random fields LMM, given the dynamics in Eq.(2.20) we know
that the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\frac{\text{Cov}(dL_i(t), dL_j(t))}{\sqrt{\text{Var}(dL_i(t))\text{Var}(dL_j(t))}} = \int_{T_{i-1}}^{T_i} \int_{T_{j-1}}^{T_j} \xi_i(t,x)\xi_j(t,y)c(x,y)dxdy \sqrt{\int_{T_{i-1}}^{T_i} \int_{T_{j-1}}^{T_j} \xi_i(t,x)\xi_i(t,y)c(x,y)dxdy \int_{T_{j-1}}^{T_j} \xi_j(t,x)\xi_j(t,y)c(x,y)dxdy},$$

which means that the instantaneous correlation of forward rates depends on both the correlation structure and the instantaneous volatilities. Indeed, if we make an approximation that $\xi_k(t)$ are constant on $[T_{k-1}, T_k]$, for $k = 1, 2, ..., N$, the correlation structure of forward rates will become

$$\int_{T_{i-1}}^{T_i} \int_{T_{j-1}}^{T_j} c(x,y)dxdy \sqrt{\int_{T_{i-1}}^{T_i} \int_{T_{j-1}}^{T_j} c(x,y)dxdy \int_{T_{j-1}}^{T_j} c(x,y)dxdy}. \quad (5.4)$$

Thus $c(x,y)$ does not need to obey the two features from the historical correlation matrix. Thus the choice of correlation functional form $c(x,y)$ for random fields LIBOR market model can be simpler than that for LIBOR market model. This is also a significant advantage of random fields model. In non-random field case, the correlation matrix need to satisfy two features, decreasing when moving away from diagonal and increasing across sub-diagonal. However, in random field case, the correlation structure $c(x,y)$ does not obey the two features. This will give more freedom to choose the correlation structure and it turns out that simple function form is enough. It will save much time on calibration. Analogous to Eq.(5.2), we only need to take

$$c(x,y) = e^{-\rho|x-y|} := c(x,y). \quad (5.5)$$

The correlation is independent of current time $t$. We will show in the estimation results that even a simpler form of $c(x,y)$ can achieve better results.

### 5.1.4 Model specification.
In this section, we provide the closed-form Black implied volatility for caplets and swaptions, in the form of the instantaneous volatility
\( \xi(t, T) \) and the correlation structure \( c(u, v) \). While extended to multi-curve framework, the model also needs the same inputs for other rates which are modeled simultaneously with LIBOR rates. Thus in multi-curve framework the number of inputs may increase according to the number of curves used and the stochastic volatility models may be more complicated.

We may need to specify parametric forms for LIBOR rate \( L_k(t) \) and FRA rates \( F^f_k(t) \), the instantaneous volatility functions \( \xi_k(t), \eta_k(t) \) and the correlation structure \( c_d(T_i, T_j), c_f(T_i, T_j), c_{df}(T_i, T_j) \). The functional form of the instantaneous volatility should satisfy the criteria of desired volatility. As shown in Sec.5.1.2, we can take \( \xi_k(t, T_k) = f(t, T_k)h(t)g(T_k) \) and \( \eta_k(t, T_k) = e(t, T_k)h(t)g(T_k) \), while for lognormal mixture model we take \( \nu^k(t, T_k) = f_i(t, T_k)h(t)g(T_k) \). \( h(t), g(T_k) \) are equal to 1 and \( f(t, T_k) \) is given as Eq.(5.1). The choice of correlation functional forms for two-curve random field LIBOR market model is the same as that for random field LIBOR market model. This means that it also shares the advantages of random fields model. Analogous to Eq.(5.2), we only need to take

\[
\begin{align*}
  c_d(x, y) &= e^{-\rho_{\infty,d}|x-y|} := c_d(x, y); \\
  c_f(x, y) &= e^{-\rho_{\infty,f}|x-y|} := c_f(x, y); \\
  c_{df}(x, y) &= e^{-\rho_{\infty,df}|x-y|} := c_{df}(x, y).
\end{align*}
\]

The correlation is independent of current time \( t \).

The Black implied volatilities for caplets and swaptions are derived as follows. For LMM, from Eq.(1.24), we have that

\[
\sigma_k^{\text{Black}}(t) = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} f^2(s, T_k)ds},
\]
and from Eq.(1.28), we know that

$$\sigma_{i,j}^{Black}(t) = \sqrt{\frac{1}{T_i - t} \sum_{k,l=1}^{j} \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \rho_{kl}(s) f(s, T_k) f(s, T_i) ds}$$

$$= \sqrt{\frac{1}{T_i - t} \sum_{k,l=1}^{j} \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \sigma_k^{Black} \sigma_t^{Black} \Theta_{t,k}(t) \rho_{k,l} \, (5.10)}$$

where

$$\Theta_{t,k}(t) = \frac{\sqrt{T_i - t} \sqrt{T_i - t} \int_t^{T_i} f(s, T_k) f(s, T_i) ds}{\int_t^{T_i} f^2(s, T_k) ds \int_t^{T_i} f^2(s, T_i) ds} \, (5.11)$$

For random field LMM, from Eq.(2.23), we have that

$$\sigma_k^{Black,RF} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \int_{T_{k-1}}^{T_k} f(s, T_k) f(s, T_i) c(x, y) dxdy ds}$$

To make the calibration easier and make comparison of Brownian motion and random field LMM, it is reasonable to set \(f(t, x) = f(t, T_k)\) for \(x \in [T_{k-1}, T_k]\). Thus the above equation becomes

$$\sigma_k^{Black,RF} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \int_{T_{k-1}}^{T_k} \tilde{\delta}_k f^2(s, T_k) ds \int_{T_{k-1}}^{T_k} c(x, y) dxdy}$$

$$= \sqrt{\frac{1}{T_{k-1} - t} c_{k,k} \int_t^{T_{k-1}} \tilde{\delta}_k f^2(s, T_k) ds} \, (5.12)$$

where \(c_{k,k} = \int_t^{T_k} \int_{T_{k-1}}^{T_k} c(x, y) dxdy\). From Eq.(2.24) and Eq.(5.5), we know that correlation structure is independent of current time \(t\), thus we have

$$\sigma_{i,j}^{Black,RF}$$

$$= \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \frac{\sum_{k,l=1}^{j} \delta_k L_k(t) \gamma_k^{i,j}(t) \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} c_{kl} \delta_k \delta_l f(s, T_k) f(s, T_i) ds}{1 + \delta_k L_k(t) 1 + \delta_l L_l(t)}}$$

$$= \sqrt{\frac{1}{T_i - t} \sum_{k,l=1}^{j} \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \sigma_k^{Black,RF} \sigma_t^{Black,RF} c_{kl}^{Black,RF} \Theta_{t,k}(t), \, (5.13)$$
where \( \Theta_{t,k}(t) \) is given in Eq.(5.11). The second equation above is obtained by using standard freezing approximation techniques, i.e., approximately evaluating the LIBOR rates \( L_k(s), t \leq s \leq T, \) appearing in the instantaneous volatility at initial time \( t. \)

For RFLMM, from Eq.(4.9), we have that

\[
\sigma_k^{Black,MR} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \left[ \int_{T_k}^{T_{k-1}} \int_{T_{k-1}}^{T_k} e(s, y)c_f(x, y)dxdy \right]ds.}
\]

To make the calibration easier and make comparison of Brownian motion and random field LMM, it is reasonable to set \( e(t, x) = e(t, T_k) \) for \( x \in [T_{k-1}, T_k]. \) Thus the above equation becomes

\[
\sigma_k^{Black,MR} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \delta_k^2 e^2(s, T_k)ds \int_{T_k}^{T_{k-1}} \int_{T_{k-1}}^{T_k} c_f(x, y)dxdy}
\]

\[
= \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} c_{f,kk} \int_{T_k}^{T_{k-1}} \delta_k^2 e^2(s, T_k)ds,}
\]

(5.14)

where \( c_{f,kk} = \int_{T_k}^{T_{k-1}} \int_{T_{k-1}}^{T_k} c_f(x, y)dxdy \) and \( c_{d,kk} = \int_{T_k}^{T_{k-1}} \int_{T_{k-1}}^{T_k} c_d(x, y)dxdy. \)

Since the correlation structure is independent of current time \( t, \) from Eq.(4.10) we have

\[
\tilde{\sigma}_{i,j}^{Black,MR}(t) = \sqrt{\frac{1}{T_{k-1} - t} \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \int_{T_k}^{T_{k-1}} \int_{T_l}^{T_{l-1}} \left[ \alpha_{ik}^{i,j}(s) \xi_k(s, u)\alpha_{i,j}^{i,j}(s)\xi_l(s, v)c_d(s, u, v) + \beta_{ik}^{i,j}(s)\eta_k(s, u)\beta_{i,j}^{i,j}(s)\eta_l(s, v)c_f(s, u, v) \right.}
\]

\[
\left. + \alpha_{ik}^{i,j}(s)\xi_k(s, u)\beta_{i,j}^{i,j}(s)\eta_l(s, v)c_d(s, u, v) \right]dudv}ds
\]

\[
= \sqrt{\frac{1}{T_{k-1} - t} \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \left\{ \delta_k \delta_l \left[ \alpha_{ik}^{i,j}(s)f(s, T_k) \right. \right.
\]

\[
\left. + \alpha_{ik}^{i,j}(s)f(s, T_l)c_{d,kl} + \beta_{ik}^{i,j}(s)e(s, T_k)\beta_{i,j}^{i,j}(s)e(s, T_l)c_{f,kl} + \alpha_{ik}^{i,j}(s)f(s, T_k) \right. \right.
\]

\[
\left. \beta_{i,j}^{i,j}(s)e(s, T_l)c_{df,kl} + \beta_{ik}^{i,j}(s)e(s, T_k)\alpha_{i,j}^{i,j}(s)f(s, T_l)c_{df,kl} \right]dudv}ds,}
\]

(5.15)
where \( c_{i,kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c_i(x,y) dx dy \), for \( i = d, f, df \).

We have two ways of calibration. One is using caplet volatilities and minimizing the mean square distance

\[
RMScapletsurf = \sqrt{\frac{1}{NN_0} \sum_{i=1}^{N} \sum_{j=1}^{N_0} \left( \frac{\sigma_{\text{Black},RMF}(m_i) - \sigma_{\text{Market}}^{i,j}}{\sigma_{\text{Black},MRF}(m_i)} \right)^2},
\]

(5.16)

where \( N \) is the number of tenor days and \( N_0 \) is the number of caplet strike price used in calibration, with non-available data treated as zero in the sum. The other one is using swaption volatility and minimizing the mean square distance

\[
RMSSwaption = \sqrt{\frac{2}{(N-1)(N-2)} \sum_{i=1}^{M} \sum_{j=i+1}^{N} \left( \frac{\sigma_{i,j}^{\text{Black}} - \sigma_{i,j}^{\text{Market}}}{\sigma_{i,j}^{\text{Black}}} \right)^2},
\]

(5.17)

where \( N \) is the number of tenor days and \( M \) is the maximum number of swaption maturities used in calibration, with non-available data treated as zero in the sum. Calibration will be shown in Sec.5.2.

5.2 Numerical Results

In this section we present the calibration results of the four different LIBOR market models, 1) single-curve LMM, 2) single-curve RFLMM, 3) two-curve LMM, and 4) two-curve RFLMM, at several different dates (before the credit crunch, during the credit crunch and after the credit crunch). We also give some examples to show the stability and accuracy of calibration, as well as the pricing and hedging performance. The four methodologies to estimate and price are described as follows.

1. Standard single-curve approach: we use the LIBOR standard yield curve to calculate the discount factors \( P(t, T) \) and forward rates for cash flow generating. The uncertainties are modeled as Brownian motions.

2. Random field single-curve approach: the curve for discount and forward is the
same as in 1), the LIBOR standard yield curve, while the uncertainties are modeled as Brownian fields.

3. Standard two-curve approach: we still use the LIBOR standard yield curve to calculate the discount factors $P(t, T)$ and use LIBOR 6M curve to calculate the forward rates, since the cap/floor and swaptions we are considering have tenor 6-month. The uncertainties are modeled as Brownian motions.

4. Random field two-curve approach: the curves used as discount and forward curves are the same as in 3). The uncertainties are modeled as random fields.

The input of the calibration is consisting of market closing prices on Sept. 12, 2005, Sept. 30, 2008, and Oct. 20, 2011 from Bloomberg, which are corresponding to the dates before/during/after the credit crisis. The market data are annualized initial forward rate curve, annualized caplet volatilities and swaption volatilities.

5.2.1 Parameter calibration. 1) First, we investigate the caps surface calibration using lognormal mixture model. The caps considered here have floating leg 6M LIBOR, maturity dates from 1 year to 30 years and strikes from 1% to 10%. Table 5.1 shows the calibration results in details. The column “RMSPE” represents the root mean square percentage error of the model volatilities and market quoted volatilities. The column “Time” (elapsed time) represents the time used to get the calibration results. The column “Range” and “Std” represent the range and standard deviation of RMSPE in each model.

From Table 5.1 we see that the single-curve model (both Brownian motion and random fields models) performed quite well before the credit crisis but did very badly during/after the credit crisis (the error accuracy degraded around 3-4 times). The reason may be that during/after the credit crisis the rates used for discounting and future cash flows generating have non-negligible spreads. The two-curve models
Table 5.1. Statistics Summary of Calibration Results for Cap Surface

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<tbody>
<tr>
<td></td>
<td>RMSPE(%)</td>
<td>Time</td>
<td>RMSPE(%)</td>
<td>Time</td>
<td>RMSPE</td>
<td>Time</td>
</tr>
<tr>
<td>single-curve LMM</td>
<td>0.83</td>
<td>2.52×10^3s</td>
<td>3.22</td>
<td>2.52×10^3s</td>
<td>3.52</td>
<td>2.52×10^3s</td>
</tr>
<tr>
<td>two-curve LMM</td>
<td>0.82</td>
<td>1.28×10^4s</td>
<td>2.54</td>
<td>1.28×10^4s</td>
<td>3.17</td>
<td>1.28×10^4s</td>
</tr>
<tr>
<td>single-curve RFLMM</td>
<td>0.51</td>
<td>7.23×10^3s</td>
<td>1.52</td>
<td>7.23×10^3s</td>
<td>2.64</td>
<td>7.23×10^3s</td>
</tr>
<tr>
<td>two-curve RFLMM</td>
<td>0.54</td>
<td>3.12×10^4s</td>
<td>1.32</td>
<td>3.12×10^4s</td>
<td>1.22</td>
<td>3.12×10^4s</td>
</tr>
</tbody>
</table>

(both Brownian motion and random fields models) performed pretty well on three dates (before/during/after the credit crisis). The reason may be that the forward curve we used (LIBOR 6M curve) has the same tenors (6M) with the caps.

Another advantage is given by the random fields setting. From Table 5.1 we notice that the models containing random fields setting have less RMSPE, compared to the Brownian motion cases. The reason is that the calibration of caplets in LMM does not contain the correlation structure, while the random fields setting uses correlation structure as one critical input. This makes the caplets calibration more accurate in random fields cases. However the price we pay is that it takes more time to get the results.

We plot the calibrated volatility smile for maturity 1 year, 3 year and 15 year in Figure 5.1. From the figure we know that when the shape of volatility smile is smooth, the original lognormal mixture model and the one with random fields setting have almost the same fitting capacity. When the shape is steep or clearly asymmetric, the lognormal mixture model with random fields setting has better fitting, while taking more time.

2) Second, we consider the joint calibration of swaption and caplets. The swap-
Figure 5.1. Examples of Caplet Volatility Smile Calibration for Maturity 1 year, 3 years, 15 years.
The considerations considered here are based on the swaps which have the same tenors as in caps discussed above, with maturity and length not exceeding 10 years. Table 5.2 shows the calibration results in details.

<table>
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<tbody>
<tr>
<td>RMSPE(%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>single-curve LMM</td>
<td>1.12</td>
<td>3.27</td>
<td>3.95</td>
</tr>
<tr>
<td></td>
<td>$7.27 \times 10^2$s</td>
<td>$7.27 \times 10^2$s</td>
<td>$7.27 \times 10^2$s</td>
</tr>
<tr>
<td>two-curve LMM</td>
<td>1.05</td>
<td>2.84</td>
<td>2.87</td>
</tr>
<tr>
<td></td>
<td>$7.34 \times 10^3$s</td>
<td>$7.34 \times 10^3$s</td>
<td>$7.34 \times 10^3$s</td>
</tr>
<tr>
<td>single-curve RFLMM</td>
<td>0.95</td>
<td>1.45</td>
<td>1.87</td>
</tr>
<tr>
<td></td>
<td>$5.37 \times 10^2$s</td>
<td>$5.37 \times 10^2$s</td>
<td>$5.37 \times 10^2$s</td>
</tr>
<tr>
<td>two-curve RFLMM</td>
<td>0.85</td>
<td>1.12</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>$5.12 \times 10^3$s</td>
<td>$5.12 \times 10^3$s</td>
<td>$5.12 \times 10^3$s</td>
</tr>
<tr>
<td>Std(%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Range(%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>single-curve LMM</td>
<td>0.32</td>
<td>2.34</td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td>0.84-1.35</td>
<td>1.15-7.5</td>
<td>1.91-6.2</td>
</tr>
<tr>
<td>two-curve LMM</td>
<td>0.55</td>
<td>1.49</td>
<td>1.56</td>
</tr>
<tr>
<td></td>
<td>0.75-1.42</td>
<td>1.75-5.4</td>
<td>1.34-4.7</td>
</tr>
<tr>
<td>single-curve RFLMM</td>
<td>0.33</td>
<td>1.43</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>0.57-1.32</td>
<td>0.82-2.25</td>
<td>0.95-2.57</td>
</tr>
<tr>
<td>two-curve RFLMM</td>
<td>0.39</td>
<td>1.21</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td>0.87-1.39</td>
<td>0.94-1.95</td>
<td>0.87-2.85</td>
</tr>
</tbody>
</table>

In Table 5.2, we have the same observations as in Table 5.1. We see that the two-curve models have better results than single-curve models (both Brownian motion and random field cases). The advantage of random fields models over Brownian models is that they take nearly half time to reach almost the same accuracy. The reason is that when the data is very big, the calibration of random fields models are more rapid since they take simpler correlation forms.

5.2.2 Pricing performance. In this section we use the market data as inputs to calibrate the parameters and then use the parameters to produce the price of the same derivatives after some time horizons. The performance of the models depend on the RMSPEs of market prices and model-implied prices. We use the market data on several days before/during/after the credit crisis. For example during the credit crisis we take some milestone days such as March 17, 2008 when J.P. Morgan Chase offered to acquire Bear Stearns, September 14, 2008 when Merrill Lynch was sold to Bank of America, September 15 when Lehman Brothers filed for bankruptcy protection, September 17, 2008 when American International Group (AIG) received $85 billion
from the U.S. Federal Reserve to avoid bankruptcy. September 25 when Washington Mutual was seized by Federal Deposit Insurance Corporation and its banking assets were sold to J.P. Morgan Chase, and October 14, 2008 when $250 billion of public money was injected into the U.S. banking system. We take the time horizon to be one day, i.e. we use yesterday’s calibration results and today’s term structure to price today’s price, then compare it to the price observed at market. The RMSPEs are the pricing errors which are shown in Table 5.3.

Table 5.3 shows that random fields models exhibit much higher accuracy than Brownian motion models on pricing, especially during the credit crunch. Before the credit crunch all four models have significantly small pricing errors, while the accuracies decrease during/after the credit crunch. On the other hand, the two-curve models also have not significantly smaller pricing errors than single-curve models.

<table>
<thead>
<tr>
<th>Days</th>
<th>RMSPE of European Swaptions after 1 day(%)</th>
</tr>
</thead>
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CHAPTER 6
ESTIMATION

In this chapter we estimate four models, single-curve LMM, single-curve R-FLMM, multi-curve LMM and multi-curve RFLMM on a series of market forward rates. First, in Sec.6.1 we review the history of the estimation of term structure models. Second, we introduce the Kalman filter, as well as unscented Kalman filter used for estimation in Sec.6.2. Third, we compare the estimation results of the four models in Sec.6.3, including the pricing and hedging performance.

6.1 Estimation of Dynamics of Term Structure Models

This section provides an overview of methods for parameter estimation in dynamic models of the term structure with discretely observed data, which in many literature take the form of stochastic differential equations (SDEs). Here we review three common techniques in term structure models estimation.

The first is the maximum-likelihood estimation (MLE), which is a method of estimating the parameters of a statistical model. Given a set of data and underlying model, the values of model parameters that the maximum likelihood approach selects will generate a distribution that gives the observed data the greatest probability(i.e., parameters that maximize the likelihood function). The second are the moments based methods, which are computing moments conditions for SDEs. There are two main approaches, generalized method of moments (GMM) and efficient method of moments (EMM). The third one is a variant of the Kalman filter, which has been a popular tool to solve estimation problems for term structure models.

The maximum likelihood estimator maximizes the likelihood function. Pearson and Sun [61] and Chen and Scott [16] first use this approach when there is a one-to-one mapping between the observation variables and state equation factors. In
this case we can assume that there are no noises on the measurements. Actually problem of using maximum likelihood is that when we have more observations than the factors of state variable the one to one mapping fails. Commonly we can estimate the model using maximum likelihood method because the likelihood function can be expressed in terms of the closed-form solutions of the Kolmogorov forward or backward equations. However, there are a few special cases that the closed-form solutions can be written and thus the numerically methods must be applied. A natural solution is to optimize the likelihood function of the discrete version SDEs, for example Euler scheme. However, the density function implied by Euler scheme provides a poor approximation to the true density.

Moments based methods are most widely used when maximum likelihood is infeasible, due to its tractability and the same asymptotic efficiency as maximum likelihood. The GMM attributable to Hansen [30] can be applied to SDEs by computing moment conditions from a discrete version. The EMM developed by Gallant and Tauchen [24] and elaborated in a series of other papers is a natural progression from GMM. The main advantage of GMM is that it does not require full density but only certain moments. Thus GMM doesn’t not make efficient use of all the information in the sample and will lead to a loss of efficiency. EMM is a natural extension of GMM that is much more efficient. However, it can not handle neither measurement noise nor non-stationary data. It is shown in Duffee and Stanton [20] that the performance of EMM with semi-nonparametric term structure model is unacceptable in even the simplest uncorrelated noises term structure settings.

Filtering methods provide an appropriate solution to the full state and parameter estimation problem. A popular tool to solve filtering problems is the Kalman filter, which was introduced in Kalman [44] and Kalman et. al [45]. The Kalman filter is a reasonable choice since it can estimate the models driven by a great variety
of randomness and can update consequently as new observations coming out. For details of parameter estimation in dynamic models of the term structure, see Duffee and Stanton [20]. In this Chapter we use unscented Kalman filter, which is further discussed in the following section, in order to estimate the parameters of the models derived in this thesis, including pricing and hedging performance investigation, given time series of discrete observations.

6.2 The Kalman Filter

Consider a probability space \((\Omega, \Sigma, P)\) and suppose that the real valued observation variable \(y_t\) is a function of a state variable \(x_t\) and a measurement noise \(\epsilon_t\)

\[
y_t = y(x_t) + \epsilon_t, \tag{6.1}
\]

while the state variable \(x_t: \Omega \rightarrow \mathbb{R}^n\) satisfies the state equation

\[
dx_t = \mu(x_t, S)dt + \sigma(x_t, S)dz_t, \tag{6.2}
\]

where \(S\) is the parameter set that contains the parameters of state equation. It is assumed that the drift and diffusion functions \(\mu\) and \(\sigma\) satisfy sufficient regularity conditions that ensure the existence and uniqueness of strong solutions. We introduce the conditional probability density (transition function) \(p(x_t|x_{t-1}, S)\), where \(S\) is the true parameter set and \(y \mapsto p(y|x, S)\) is the probability density function of \(x_t\) given \(x = x_{t-1}\). It can be shown that \(p\) is the solution to Kolmogorov backward or forward equation. The Markov property of state equation makes it possible to express the likelihood function in terms of the transition density function, i.e.,

\[
\text{ML}(S) = p(x_0|S) \prod_{t=1}^{T} p(x_t|x_{t-1}, S). \tag{6.3}
\]

The primary difficulty in estimating \(S\) is that the exact functional form for the transition density function is often unknown or intractable.
The filtering problem is the following: Given the observation $y_s$ for $0 \leq s \leq t$ satisfying Eq.(6.1), what is the best estimate $\hat{x}_t$ of the true state $x_t$ of the system Eq.(6.2) based on these observation? In this chapter we will use the Unscented Kalman Filter to estimate the true states for our models and use MLE to estimate the parameters. The original Kalman filter is used to estimate a linear dynamic system given observations perturbed by noise. However, the method can not deal with case of nonlinear SDEs. Thus several extended versions were introduced, such as Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF). The EKF is discussed in detail in Anderson and Moore [7] and Jazwinski [41]. This method adapts linearization techniques, namely multivariate Taylor Series expansions to linearize nonlinear transition function and observation function. This method is desirable if the functions are not too highly nonlinear, since it is easy to implement. However, if the function are highly nonlinear, the linearization will cause huge error which will influence the estimation and the result is not accurate. The UKF was introduced by Julier and Uhlman [43]. The UKF uses the statistical linearization techniques, which linearize a nonlinear function of a random variable through a linear regression between $n$ points drawn from the prior distribution of the random variable ($n$ is the dimension of state variable). In the UKF, the probability density is approximated by a deterministic sampling called unscented transformation which is designed to propagate the mean and covariance of the state variable and observation through nonlinear transformation and then to build the prior/posterior estimate and corresponding error. When the models are nonlinear and not suitable to be linearized, this method can produce much better and more stable estimate of the state variable than EKF. For simplicity, the algorithm of UKF are given below without proof. More details of this method can be found in Julier and Uhlman [43] for additive noise case and Wan and Van de moire [69] for non-additive noise case.

Suppose the observation equation and the transition equation expressing the
discrete-time evolution of state variable are given as in the following form

\[ y_t = H(x_t, S, \epsilon_t); \quad (6.4) \]
\[ x_{t+1} = G(x_t, S, \nu_{t+1}); \quad (6.5) \]

with moments

\[ E(\epsilon_t) = 0; \quad E(\epsilon_t^T \epsilon_t) = R_t(S); \quad E(\nu_{t+1}) = 0; \quad E(\nu_{t+1}^T \nu_t) = Q_t(S) \quad \text{and} \quad E(\epsilon_t \nu_s) = 0, \]

where \( H \) and \( G \) are generally nonlinear functions of \( x_t \). We assume that \( \epsilon_t \) and \( \nu_s \) are both following Gaussian distribution and uncorrelated for all \( s, t > 0 \). Notice that the innovation process \( \nu_{t+1} \) and measurement noise may not need to be additive in general.

Now we will review the unscented Kalman filter first and then provide the algorithm without proof. Since the innovation process and the measurement noise are in general non-additive (as the same case in our model), the scheme is thus applied to augmented state \( x^a_t = [x^T_t, \epsilon^T_t, \nu^T_{t+1}]^T \) with length \( L \). One-step-ahead prediction (priori estimate) of the augmented state variable \( x^a_t \) at time \( t \) and the observations are denoted \( \hat{x}_{t+1|t} \) and \( \hat{y}_{t+1|t} \). The variance and the covariance matrices of these prediction given observations up to time \( t \) are denoted \( P_{t+1|t}, V_{t+1|t}, \) and \( U_{t+1|t} \) respectively. The prediction (posterior estimate) of the augmented state variable \( x^a_t \) at time \( t \) given the observations up to time \( t + 1 \) and its associated variance-covariance matrix are denoted \( \hat{x}_{t+1|t+1} \) and \( P_{t+1|t+1} \) respectively.

The unscented Kalman filter basically contains three steps, unscented transformation step, prediction step and update step. The unscented transformation step takes a set of appropriated chosen sample points with weights to recover the mean and covariance after the nonlinear function action. The sample points \( x^i_{t|t} \) with weights
$\omega^i$ are chosen as follows

\[
\begin{align*}
x^0_{t\mid t} &= \hat{x}_{t\mid t}; \\
x^i_{t\mid t} &= \hat{x}_{t\mid t} + \left( \sqrt{\frac{L}{1-\omega^0}} P_{t\mid t} \right) \omega^i; \quad i = 1, 2, ..., L; \\
x^{i+L}_{t\mid t} &= \hat{x}_{t\mid t} - \left( \sqrt{\frac{L}{1-\omega^0}} P_{t\mid t} \right) \omega^i; \quad i = 1, 2, ..., L.
\end{align*}
\]

where $\left( \sqrt{\frac{L}{1-\omega^0}} P_{t\mid t} \right) i$ is the i-th column of the matrix square root of $\sqrt{\frac{L}{1-\omega^0}} P_{t\mid t}$. We need to specify the $\omega^0$, which has requirement $-1 \leq \omega^0 \leq 1$ and $\sum_{i=0}^{2^n} \omega^i = 1$. Actually, $\omega^i$ controls the position of the sample points: $\omega^0 \geq 0$ points trend to move further from the origin, $\omega^0 \leq 0$ tend to be closer to the origin. With the such setting, it is proven that the mean and covariance can be recovered up to the second order. Thus if the distribution of the random variable can be determined by the first two moments like normal distribution, using unscented transformation, we can capture all the distribution features of the random variable.

The prediction step calculates $\hat{x}_{t+1\mid t}$, $\hat{y}_{t+1\mid t}$ and $P_{t+1\mid t}$, $V_{t+1\mid t}$, $U_{t+1\mid t}$, the prediction of the state variable, observation and their associated variance-covariance matrices based on the estimate of state variable at the previous step. Using the sample points, the prediction equations are given as follows

\[
\begin{align*}
\hat{x}_{t+1\mid t} &= \sum_{i=0}^{2L} \omega^i G(x^i_{t\mid t}); \\
P_{t+1\mid t} &= \sum_{i=0}^{2L} \omega^i [G(x^i_{t\mid t}) - \hat{x}_{t+1\mid t}][G(x^i_{t\mid t}) - \hat{x}_{t+1\mid t}]^T; \\
\hat{y}_{t+1\mid t} &= \sum_{i=0}^{2L} \omega^i H(x^i_{t\mid t}); \\
V_{t+1\mid t} &= \sum_{i=0}^{2L} \omega^i [H(x^i_{t\mid t}) - \hat{y}_{t+1\mid t}][H(x^i_{t\mid t}) - \hat{y}_{t+1\mid t}]^T; \\
U_{t+1\mid t} &= \sum_{i=0}^{2L} \omega^i [G(x^i_{t\mid t}) - \hat{x}_{t+1\mid t}][H(x^i_{t\mid t}) - \hat{y}_{t+1\mid t}]^T.
\end{align*}
\]
The update step calculates $\hat{x}_{t|t}$ and $P_{t|t}$, the prediction of the state variable from previous time step given the new observation $y_t$ for current time step. The equations are given as follows

$$
\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + K_{t+1|t}\epsilon_{t+1};
$$

$$
P_{t+1|t+1} = P_{t+1|t} - K_{t+1|t}V_{t+1|t}K_{t+1|t}^{-1}.
$$

We can notice that it is no necessary to calculate Jacobians or Hessians and the amount of the chosen sample is in the order of the dimension of state variable, which makes the implementation of UKF easier and more accurate. Furthermore, it can be proven that the overall number of computations are of the same order as the EKF.

Based on the procedure of unscented Kalman filter, Julier and Uhlmann [43] provides the recursion algorithm as follows. Given the initial state $x_0$ with known mean $t \hat{x}_0 = \mathbb{E}[x_0]$ and variance-covariance matrix $P_0 = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$, the unscented Kalman filter recursion begins with the mean and variance of augmented state variable $x_0^2$: $\hat{x}_0|0 = \mathbb{E}[x_0^0] = [\hat{x}_0^T, 0, 0]^T$, $P_{0|0} = \mathbb{E}[(x_0^0 - \hat{x}_0|0)(x_0^0 - \hat{x}_0|0)^T] = \text{diag}\{P_0, Q_0, R_0\}$ (Step 0). Generally the recursion steps are as follows:

1. Given $\hat{x}_{t|t}$ and $P_{t|t}$, compute sample points $x_{i|t}^0, x_{i|t}^i, x_{i|t}^{i+L}$ for $i = 1, ..., L$:

   $$
x_{i|t}^0 = \hat{x}_{t|t}; x_{i|t}^i = \hat{x}_{t|t} + (\sqrt{\frac{L}{1-\lambda}}P_{t|t})_{ii}; x_{i|t}^{i+L} = \hat{x}_{t|t} - (\sqrt{\frac{L}{1-\lambda}}P_{t|t})_{ii},$$

   where $(\sqrt{\frac{L}{1-\lambda}}P_{t|t})_{ii}$ is the $i$-th column of the matrix square root of $\sqrt{\frac{L}{1-\lambda}}P_{t|t}$ and $\omega^i = \frac{1-\lambda}{2L}$ for $i = 1, ..., 2L$.

2. Compute the one-period-ahead prediction and variance of $x_{t+1}^a$:

   $$
   \hat{x}_{t+1|t} = \sum_{i=0}^{2L} \omega^i G(x_{i|t}^i) \text{ and } P_{t+1|t} = \sum_{i=0}^{2L} \omega^i [G(x_{i|t}^i) - \hat{x}_{t+1|t}][G(x_{i|t}^i) - \hat{x}_{t+1|t}]^T.
   $$

3. Compute the one-period-ahead prediction and variance of $y_{t+1}$, covariance of $x_{t+1}^a$ and $y_{t+1}$:

   $$
   \hat{y}_{t+1|t} = \sum_{i=0}^{2L} \omega^i H(x_{i|t}^i), \quad V_{t+1|t} = \sum_{i=0}^{2L} \omega^i [H(x_{i|t}^i) - \hat{y}_{t+1|t}][H(x_{i|t}^i) - \hat{y}_{t+1|t}]^T, \quad \text{and}
   $$
\[ U_{t+1|t} = \sum_{i=0}^{2T} \omega^i [G(x_{i|t}) - \hat{x}_{t+1|t}] [H(x_{i|t}) - \hat{y}_{t+1|t}]^T. \]

4. Compute the prediction error in \( y_{t+1} \): \( e_{t+1} = y_{t+1} - \hat{y}_{t+1|t} \).

5. Compute the Kalman gain: \( K_{t+1|t} = U_{t+1|t} V_{t+1|t}^{-1} \).

6. Update the prediction and variance of \( x_{t+1} \):
   \[ \hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + K_{t+1|t} e_{t+1} \quad \text{and} \quad P_{t+1|t+1} = P_{t+1|t} - K_{t+1|t} V_{t+1|t} K_{t+1|t}^{-1}. \]

Once we get the time series of \( e_t \) and \( V_{t-1|t} \), we can write down the transition density function \( p(x_t|x_{t-1}, S) \). Given \( S \) and \( x_{t-1}, x_t \) has multivariate normal distribution with determinate mean and covariance matrix. Thus we can use the Maximum Likelihood Estimate (MLE) to get the estimate. The best estimate of \( S \) is the value that maximize the log-likelihood function

\[
\text{LogML}(S) = \ln p(x_T|S) = \ln p(x_0|S) + \sum_{t=1}^{T} \ln p(x_t|x_{t-1}, S),
\]

which is equivalent to maximizing

\[
-\frac{1}{2} \sum_{t=1}^{N} [N \ln(2\pi) + \ln |V_{t-1}| + e_t^T V_{t-1}^{-1} e_t]. \quad (6.6)
\]

Suppose \( \hat{S} \) is the parameter set that maximizes the log-likelihood function Eq.(6.6), i.e.,

\[
\hat{S} = \arg \max_S \text{LogML}(S), \quad (6.7)
\]

and \( S_0 \) is the real parameter set. The variance covariance matrix of \( \hat{S} \) can measure the accuracy of the maximum likelihood estimate \( \hat{S} \). An estimate of the matrix is based on the outer product of first derivatives of the log-likelihood function

\[
\text{E}[(\hat{S} - S_0)(\hat{S} - S_0)^T] \approx \frac{1}{T^2} \sum_{t=1}^{T} \left( \frac{\partial \text{LogML}}{\partial S} \right) \times \left( \frac{\partial \text{LogML}}{\partial S} \right) |_{S = \hat{S}}, \quad (6.8)
\]

equivalently for a single parameter \( s \in S \), the standard error can be approximated by

\[
\left[ -\frac{\partial^2 \text{LogML}(s)}{\partial s^2} \right]^{-1}. \quad (6.9)
\]
More details about this result can be found in Hamilton [29]. Following the Kalman filter recursion steps 1-5, we can calculate the \( e_t \) and \( V_{t|t-1} \) needed for transition density functions and use the maximum likelihood estimation methods to find the best estimate of the parameter set which maximizes formula (6.6).

6.2.1 Model Specification.

6.2.1.1 Random fields models. The two-curve random field LIBOR market dynamics were exhibited in Eq.(4.8), where the odd lines shows the dynamics of \( L_k(t) \) in discounting curve, which can also be regarded as the dynamics of random fields model in one-curve case. Thus we first write down the discrete observation equation for \( L_k(t) \) in the discounting curve, which is also the one for one-curve model and then extend to \( F^F_k(t) \) in the forwarding curve. We now take the case \( j < k \) as example and the cases of \( j < k \) and \( j = k \) are analogous.

Under the \( T_j \)-forward measure with \( j < k \), the Euler discretization of dynamics of \( L_k(t) \)

\[
dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^T_j(t, u) + \Lambda^k_j(t, u)dt]du, \tag{6.10}
\]

with

\[
\Lambda^k_j(t, u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, v) c_d(v, u)}{\delta_i L_i(t) + 1} dv,
\]

is given as

\[
\frac{L_k(t_{n+1}) - L_k(t_n)}{L_k(t_n)} = \sum_{i=j+1}^{k} A_n(i, k) \frac{\delta_i L_i(t_n)}{\delta_i L_i(t_n) + 1} \Delta t + \sqrt{\Delta t} z_k, \tag{6.11}
\]

where \( \Delta t = t_{n+1} - t_n, \ z \sim N(0, A_n) \), and the covariance matrix \( A_n \) has entries

\[
A_n(i, k) = \int_{T_{i-1}}^{T_i} \int_{T_{k-1}}^{T_k} \xi_k(t, u) \xi_i(t, v) c_d(u, v) dudv.
\]

The discrete transition equation thus can be written as

\[
x_{n+1} = x_n + M_n + \text{diag}(x_n) z \sqrt{\Delta t}, \tag{6.12}
\]
with $M_n$ a vector with entries $M_n(k) = x_{n-1}(k) \sum_{i=j+1}^{k} A_{n-1}(i, k) \frac{\delta_{i, x_{n-1}(i+1)}}{\delta_i} \Delta t$, where $x_{n-1}(k)$ is the $k$-th entry of state vector $x_{n-1}$. It corresponds to choose $G(x_t, S) = x_t + M_n + \text{diag}(x_t) \nu_{t+1} \sqrt{\Delta t}$ and $z = \nu_{t+1}$ in Eq. (6.1), which means that $Q_t = A_n$, where diag($x_t$) is the diagonal matrix with $x_t$ on the diagonal.

The goal of filtering is to estimate the correlation $c_d(t, u, v)$ and the instantaneous volatility $\xi_k(t, u)$. We take the parametric form as in Sec.5.1.2:

$$\xi_k(t, T_k) = [a + b(T_k - t)]e^{-c(T_k - t)} + d,$$

and

$$c_d(t, u, v) = e^{-\frac{\rho_{\infty,d}}{\infty-1}|u-v|}.$$  

Thus the parameter set in the model is $S = \{a, b, c, d, \rho_{\infty,d}\}$. Notice that $S$ is contained only in $A_n$.

Combining the discrete transition equation we derived above to the observation equation we will cover in later literature and following the Kalman filter recursion steps 1-5, we can calculate the $e_t$ and $V_{t|t-1}$ for transition density functions and use the maximum likelihood estimation methods to find the best estimate of the parameter set which maximizes formula (6.6). The above procedure plays the filtering steps for discounting curve in two-curve random field LIBOR market model, which is also the procedure for one-curve case. The same procedure applies to FRA rates $F^j_k(t)$ for forwarding curve in two-curve models.

Under the $T_j$-forward measure with $j < k$, the Euler discretization of dynamics of $F^j_k(t)$

$$dF^j_k(t) = F^j_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u)[dB^j(t, u) + \Lambda^j_k(t, u)dt]du,$$

with

$$\Lambda^j_k(t, u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \delta_i L_i(t) \xi_i(t, v)c_{dj}(v, u) \delta_i L_i(t) + 1 dv,$$
is given as
\[
\frac{F_k^{f}(t_{n+1}) - F_k^{f}(t_n)}{F_k^{f}(t_n)} = \sum_{i=j+1}^{k} C_n(i, k) \frac{\delta_i L_i(t_n)}{\delta_i L_i(t_n)} + \Delta t + \sqrt{\Delta t} w,
\]
where \(C_n(i, k) = \int_{T_{i-1}}^{T_i} \int_{T_{k-1}}^{T_k} \eta_k(t, u) \xi_i(t, v) c_{df}(u, v) dudv, w \sim \mathcal{N}(0, B_n)\) and the covariance matrix \(B_n\) has entries \(B_n(i, k) = \int_{T_{i-1}}^{T_i} \int_{T_{k-1}}^{T_k} \eta_k(t, u) \eta_k(t, v) c_f(t, u, v) dudv.\)

The discrete transition equation thus can be written as
\[
x_{n+1} = x_n + M_n' + \text{diag}(x_n) z \sqrt{\Delta t},
\]
with \(M_n'\) a vector with entries \(M_n'(k) = x_{n-1}(k) \sum_{i=j+1}^{k} C_{n-1}(i, k) \frac{\delta_i x_{n-1}(i)}{\delta_i x_{n-1}(i) + 1} \Delta t.\) It corresponds to choose \(G(x_t, S) = x_t + M_n' + \text{diag}(x_t) \nu_{t+1} \sqrt{\Delta t}\) and \(z = \nu_{t+1}\) in Eq.(6.1), which means that \(Q_t = B_n.\)

The goal is to estimate the correlation \(c_f(u, v), c_{df}(u, v)\) and the instantaneous volatility \(\eta_k(t, u).\) We take the parametric form as in Sec.5.1.3
\[
\eta_k(t, T_k) = [a' + b'(T_k - t)e^{-c'(T_k-t)}] + d',
\]
\[
c_{df}(x, y) = e^{-\rho_{\infty,df}|x-y|},
\]
and
\[
c_f(x, y) = e^{-\rho_{\infty,f}|x-y|}.
\]
We denote the set of parameters of the model as
\[
S' := \{a', b', c', d', \rho_{\infty,df}, \rho_{\infty,f}\}.
\]

6.2.1.2 Brownian motion models. For Brownian motion models, the innovation noise term is modeled by Brownian motion \(W(t).\) Following the analysis procedure in Sec. 6.2.1.1, Eq.(6.10) in Brownian motion has the form
\[
dL_k(t) = L_k(t)\xi_k(t, T_k)[dW_k^{T_j}(t) + \Lambda_k^{T_j}(t, T_k)dt],
\]
with
\[ N^k_j(t, T_k) = \sum_{i=j+1}^{k} \frac{\delta_i L_i(t) \xi_i(t, T_i) c_d(i, k)}{\delta_i L_i(t) + 1} dv. \]

Thus the Euler discretization of the dynamics has the same form as in Eq.(6.11) with
\[ A_n(i, k) = \xi_k(t_n, T_k) \xi_i(t_n, T_i) c_d(i, k). \]

As we can see, the models of innovation, random fields or Brownian motion case have only difference on \( A_n(i, k) \). Thus for two-curve models, we have
\[ B_n(i, k) = \xi_k(t_n, T_k) \eta_i(t_n, T_i) c_d(i, k), \]
\[ C_n(i, k) = \eta_k(t_n, T_k) \eta_i(t_n, T_i) c_d(i, k). \]

The parametric forms of volatility function for Brownian motion are the same for random fields models as in Sec.5.1.2, while the correlations have the form as in Eq.(5.3)
\[ c_d(x, y) = e^{-\frac{|i-j|}{N-1} \left( \rho_{\infty,d} + \rho_{0,d} \frac{N-i-j+1}{N-2} \right)}, \]
\[ c_f(x, y) = e^{-\frac{|i-j|}{N-1} \left( \rho_{\infty,f} + \rho_{0,f} \frac{N-i-j+1}{N-2} \right)}, \]
and
\[ c_{df}(x, y) = e^{-\frac{|i-j|}{N-1} \left( \rho_{\infty,df} + \rho_{0,df} \frac{N-i-j+1}{N-2} \right)}. \]

The derivation of observation equation is quite straightforward. For both random field and Brownian motion models, we assume that the measurement error follows multivariate normal distribution, i.e.,
\[ \epsilon_t \sim \mathcal{N}(0, R), \]
where \( R \) is the covariance matrix defined in Eq.(6.1). Notice that the covariance matrix \( R \) should be estimated from the market data. In this thesis, we estimate the parameters from the data set of forward rates. Analogous to the discussion in Li and Zhao [52], we can set \( R \) to be a diagonal matrix with a parameter \( R \) in diagonal. Thus we have only one parameter to be estimated, \( \sigma \).
6.3 Numerical Results

6.3.1 Estimation process of models.

In this section we present the unscented Kalman filter estimation results of the four different LIBOR market models, 1) single-curve LMM, 2) single-curve random field LMM, 3) two-curve LMM, and 4) two-curve random field LMM, give time series of market data. We also consider the pricing and hedging performance of these four models.

The dataset used for UKF estimation consists of daily forward rates from July 9, 2007–Oct. 9, 2009. The forward rates are stemmed from three different curves, LIBOR standard curve, OIS curve, LIBOR 6M curve, as described in Sec.5.1.1. The reason why we choose the time interval as 6M is that the swaption volatilities used for pricing and hedging are based on swap rates on LIBOR-6M. Analogous to the discussion in Sec.5.1.1, according to Ametrano and Bianchetti [3] and Bianchetti et al. [10], we can built up the different curves as follows.

1. LIBOR standard curve: the classic yield curve bootstrapped from short term LIBOR deposits (below 1 year), mid-term FRA on LIBOR 3M (below 2 years) and mid/long-term swaps on LIBOR 6M (after 2 years). In single-curve modeling this curve will be used for both discounting and forwarding.

2. OIS curve: the curve bootstrapped from the U.S. OIS rates. In two-curve modeling this curve will be used as discount curve.

3. LIBOR 6M curve: the LIBOR 6M curve bootstrapped from the LIBOR deposit 6M, mid-term FRA on LIBOR-6M (up to 2 years) and mid/long-term swaps on LIBOR 6M (after 2 years). In single-curve modeling this curve will be used for discounting and forwarding curve are bootstrapped using as discounting curve.
In Figure 6.1 and Figure 6.2 we plot the yield curves and the corresponding forward curve described above. The date refers to Sep. 20, 2012. Figure 6.1 shows the term structure of the LIBOR standard and U.S. OIS curve, which are used for discounting. The OIS curve is lower than LIBOR standard, due to the lower credit and liquidity risk. Figure 6.2 shows the bootstrapped LIBOR forward and the LIBOR 6M curve using monotone convex spline interpolation method. We can see that the two forward curves present very similar term structure with small difference. In addition, the two curves differs in the short term window, since up to 2-year maturity they used FRA rates based on different tensors. We show that the choice of forwarding curve has an ineligibles impact on the pricing of interest rate derivatives, even if the differences between these two curves are very small (less than 2 basis points). This result coincides the conclusion of Bianchetti and Carlicchi [10].

![Figure 6.1. Term Structure of LIBOR Standard and OIS Curves. Quotations Sep. 20, 2012 (source: Bloomberg)](image)

To compare the random fields models and Brownian motion models, as well as the single-curve models and multi-curve models, we use the following four methodologies to estimate and price.
Figure 6.2. Term Structure of LIBOR Forward and LIBOR 6M Curves. Quotations Sep. 20. 2012(source: Bloomberg)

1. Standard single-curve approach: we use the LIBOR standard yield curve to calculate the discount factors $P(t, T)$ and forward rates for cash flow generating. The uncertainties are modeled as Brownian motions.

2. Random field single-curve approach: the curve for discount and forward is the same as in 1), the LIBOR standard yield curve, while the uncertainties are modeled as Brownian fields.

3. Standard two-curve approach: we still use the LIBOR standard yield curve to calculate the discount factors $P(t, T)$ and use LIBOR 6M curve to calculate the forward rates, since the cap/floor and swaptions we are considering have tenor 6-month. The uncertainties are modeled as Brownian motions.

4. Random field two-curve approach: the curves used as discount and forward curves are the same as in 3). The uncertainties are modeled as random fields.
6.3.2 Parameter estimation. In this subsection we estimate the parameters from the time series of term structure data using the unscented Kalman filter described in last section.

For both periods, comparing the results of one-curve and two-curve estimation, we see that the discount curve and the forward curve have slightly different dynamics. This result coincides with the fact that the forward curve differs from the discount curve only in the short term window, since up to 2-year maturity it is bootstrapped using the FRA and deposit rates. Notice that RFLMM models have one less parameter in correlation structure. The reason is that in random fields models, the correlation structure is not exactly the same to the instantaneous changes of forward rates, thus a simpler functional form is enough to have the desirable features. See Sec. 5.1.3 for detail discussions.

Table 6.1 and 6.2 show the performance of the LMM and RFLMM in capturing bond yields. Given the parameters and state variables estimated based on 0.5, 1, 3, 5, 7, 10, 20 years LIBOR standard yields and OIS yields, for in-sample and out-of-sample periods, we can compute the one-day-ahead projection of yields. One-day-ahead projection is computed as follows. First, we use the market rate at day $t_1$ and estimated parameters to predict the model-implied rate at next day $t_2$, given by the unscented Kalman filter state prediction. Second, the error is computed as the difference of market rate and model-implied rate at day $t_2$. This is the one-day-ahead prediction at day $t_1$. Thus we have a time series of one-day-projection. The entries in the tables are defined as the RMSPEs of model-implied rates and market rates for different tenors, over the in-sample or out-of-sample period. From the tables, we see that for both non-random fields models and random fields models capture the yields very well, with better performance for random fields models. Not surprisingly, the in-sample errors are less than out-of-sample errors. For all models, the RMSPEs
decrease as time increase. This result coincide the results of Li and Zhao [52]. Another interesting observation is that OIS curve has less error than LIBOR standard curve. This is also reasonable, since during the credit crunch, the LIBOR curve is more oscillated than OIS curve.

Table 6.1. RMSEs of Model Predicted Changes for LIBOR standard curve(%) 

<table>
<thead>
<tr>
<th>Models</th>
<th>Overall</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMM, In-sample</td>
<td>0.018707</td>
<td>2.0450</td>
<td>0.018628</td>
<td>0.018430</td>
<td>0.018430</td>
<td>0.018430</td>
<td>0.018430</td>
<td>0.018430</td>
</tr>
<tr>
<td>RFLMM, In-sample</td>
<td>0.012640</td>
<td>0.013855</td>
<td>0.012574</td>
<td>0.012449</td>
<td>0.012449</td>
<td>0.012449</td>
<td>0.012449</td>
<td>0.012449</td>
</tr>
<tr>
<td>LMM, Out-of-sample</td>
<td>2.0948</td>
<td>2.2899</td>
<td>2.0860</td>
<td>2.0637</td>
<td>2.0637</td>
<td>2.0637</td>
<td>2.0637</td>
<td>2.0637</td>
</tr>
<tr>
<td>RFLMM, Out-of-sample</td>
<td>0.014155</td>
<td>0.015515</td>
<td>0.014080</td>
<td>0.013940</td>
<td>0.013940</td>
<td>0.013940</td>
<td>0.013940</td>
<td>0.013940</td>
</tr>
</tbody>
</table>

Table 6.2. RMSEs of Model Predicted Changes for OIS curve(%) 

<table>
<thead>
<tr>
<th>Models</th>
<th>Overall</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMM, In-sample</td>
<td>0.015805</td>
<td>0.017277</td>
<td>0.015738</td>
<td>0.015571</td>
<td>0.015570</td>
<td>0.015570</td>
<td>0.015570</td>
<td>0.015570</td>
</tr>
<tr>
<td>RFLMM, In-sample</td>
<td>0.010679</td>
<td>0.011706</td>
<td>0.010623</td>
<td>0.010518</td>
<td>0.010518</td>
<td>0.010518</td>
<td>0.010518</td>
<td>0.010518</td>
</tr>
<tr>
<td>LMM, Out-of-sample</td>
<td>0.015958</td>
<td>0.017445</td>
<td>0.015891</td>
<td>0.015722</td>
<td>0.015722</td>
<td>0.015722</td>
<td>0.015722</td>
<td>0.015722</td>
</tr>
<tr>
<td>RFLMM, Out-of-sample</td>
<td>0.010783</td>
<td>0.011819</td>
<td>0.010726</td>
<td>0.010620</td>
<td>0.010620</td>
<td>0.010620</td>
<td>0.010620</td>
<td>0.010620</td>
</tr>
</tbody>
</table>

As we have mentioned in Sec.2.2, two main advantages of random field LIBOR market modeling are that it is no necessary to specify the number of factors and re-calibrate over time. The unnessessariness of number of factors specification was demonstrated in the beginning of estimation. We directly specify the instantaneous volatility $\sigma(t,T)$ and correlation structure $c(u,v)$, which determine the covariance of instantaneous changes of forward rates.

Here we show empirically that it does not need to re-calibration for RFLM-M. Pang [60] pointed out that the covariance function of forward rates maintain similar shapes throughout a long period of time(one month in his examination), by examining the stability of the eigenvectors and eigenvalues of covariance structure over the period. In our approach, the elements of covariance can be written as $\int_{T_{i-1}}^{T_{i}} \int_{T_{j-1}}^{T_{j}} \sigma(t,u)\sigma(t,v)c(u,v)dudv$. Given the parameters $a, b, c, d$ and $\rho_{\infty}, \rho_{0}$ esti-
mated in the two periods, we can examine the stability of covariance. Figure 6.3 plots the first six eigenvectors for the two periods. The figure shows that the first six eigenvectors are stable between the two periods. Morini and Webber [57] showed that 88.5% of the variance in EUR forward rates can be explained by the first six eigenvectors/eigenvalues. Thus we can say that there is no need to re-calibrate for random fields models.

### Table 6.3. The First Six Eigenvectors of Covariance Matrix From both In-sample and Out-of-sample

<table>
<thead>
<tr>
<th>Eigenvectors</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>First, In-sample</td>
<td>-7.48E-14</td>
<td>-1.04E-13</td>
<td>-1.22E-13</td>
<td>-9.55E-108</td>
<td>-3.97E-14</td>
<td>1.23E-13</td>
<td>-1.63E-13</td>
<td>6.08E-14</td>
<td>-1.34E-12</td>
<td>-9.83E-04</td>
<td>-1.00E+00</td>
<td></td>
</tr>
<tr>
<td>First, Out-of-sample</td>
<td>-2.93E-13</td>
<td>-4.13E-13</td>
<td>-4.92E-13</td>
<td>-3.44E-105</td>
<td>-1.00E-142</td>
<td>5.09E-13</td>
<td>-4.39E-13</td>
<td>6.26E-13</td>
<td>-4.26E-12</td>
<td>-1.51E-01</td>
<td>-1.00E+00</td>
<td></td>
</tr>
<tr>
<td>Second, In-sample</td>
<td>1.21E-11</td>
<td>1.14E-10</td>
<td>1.34E-10</td>
<td>-1.49E-104</td>
<td>-1.95E-142</td>
<td>-1.63E-10</td>
<td>1.14E-10</td>
<td>-6.59E-11</td>
<td>1.47E-09</td>
<td>1.06E+00</td>
<td>-9.80E-04</td>
<td></td>
</tr>
<tr>
<td>Second, Out-of-sample</td>
<td>2.14E-10</td>
<td>3.06E-10</td>
<td>3.58E-10</td>
<td>-1.26E-101</td>
<td>3.19E-10</td>
<td>-1.90E-10</td>
<td>3.10E-09</td>
<td>1.06E+00</td>
<td>-1.51E-03</td>
<td>-9.80E-04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third, In-sample</td>
<td>-3.42E-05</td>
<td>-4.75E-05</td>
<td>5.58E-05</td>
<td>-1.90E-96</td>
<td>5.53E-13</td>
<td>5.64E-05</td>
<td>-4.73E-05</td>
<td>2.74E-05</td>
<td>1.00E+00</td>
<td>2.41E-06</td>
<td>-1.86E-10</td>
<td></td>
</tr>
<tr>
<td>Third, Out-of-sample</td>
<td>2.14E-10</td>
<td>3.06E-10</td>
<td>3.58E-10</td>
<td>-1.26E-101</td>
<td>3.19E-10</td>
<td>-1.90E-10</td>
<td>3.10E-09</td>
<td>1.06E+00</td>
<td>-1.51E-03</td>
<td>-9.80E-04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fourth, In-sample</td>
<td>5.14E-02</td>
<td>7.65E-02</td>
<td>9.48E-02</td>
<td>-2.81E-89</td>
<td>8.92E-13</td>
<td>7.05E-02</td>
<td>-2.81E-89</td>
<td>3.14E-02</td>
<td>5.85E-12</td>
<td>5.31E-17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fourth, Out-of-sample</td>
<td>6.76E-02</td>
<td>9.48E-02</td>
<td>1.13E-01</td>
<td>-3.75E-87</td>
<td>1.72E-12</td>
<td>-1.17E-01</td>
<td>1.00E-01</td>
<td>5.99E-01</td>
<td>9.72E-01</td>
<td>1.45E-11</td>
<td>1.78E-16</td>
<td></td>
</tr>
<tr>
<td>Fifth, In-sample</td>
<td>-3.80E-01</td>
<td>-4.79E-01</td>
<td>4.94E-01</td>
<td>-7.92E-82</td>
<td>4.57E-12</td>
<td>4.30E-01</td>
<td>-3.10E-01</td>
<td>1.81E-01</td>
<td>1.41E-17</td>
<td>6.65E-24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fifth, Out-of-sample</td>
<td>-3.87E-01</td>
<td>-4.82E-01</td>
<td>4.94E-01</td>
<td>-6.37E-80</td>
<td>3.11E-17</td>
<td>4.17E-01</td>
<td>-2.92E-01</td>
<td>1.47E-01</td>
<td>2.28E-01</td>
<td>5.42E-17</td>
<td>6.12E-23</td>
<td></td>
</tr>
<tr>
<td>Sixth, In-sample</td>
<td>4.95E-01</td>
<td>3.14E-01</td>
<td>-6.10E-02</td>
<td>-1.81E-74</td>
<td>-1.33E-12</td>
<td>3.96E-01</td>
<td>-5.01E-01</td>
<td>3.35E-01</td>
<td>3.39E-02</td>
<td>3.39E-23</td>
<td>1.26E-30</td>
<td></td>
</tr>
<tr>
<td>Sixth, Out-of-sample</td>
<td>4.93E-01</td>
<td>3.01E-01</td>
<td>-7.97E-02</td>
<td>-1.87E-72</td>
<td>-5.94E-10</td>
<td>4.00E-01</td>
<td>-4.99E-01</td>
<td>3.27E-01</td>
<td>5.31E-02</td>
<td>2.04E-22</td>
<td>-2.10E-29</td>
<td></td>
</tr>
</tbody>
</table>

### 6.3.3 Pricing performance.

In this section, we examine the pricing performance of both caps and swaptions for four different models using the estimation results in Sec.6.3.2. We use the estimated parameters to compute the model-implied caplet/swaption prices and thus the pricing errors of model-implied prices and market prices to compare the four different models. The pricing error is computed as the root of mean square percentage errors (RMSPEs) of the average of percentage pricing errors over all maturities and all lengths, i.e.,

\[
RMSPE = \sqrt{\frac{\sum_{\text{maturities}} \sum_{\text{lengths}} e_{i,j}^2}{M}},
\]

where \(e_{i,j}\) is the percentage error of the market caplet/swaption price and the model-implied swaption price, and \(M\) is the total number of caplets/swaptions included in calculation. The pricing procedure is described as follows. First, we estimate the models parameters using the time series of term structure from a fixed period. Second, we use the estimated parameters and the term structure from this period to compute the model-implied prices of caplets/swaptions. Third, we compute the time
Figure 6.3. The Eigenvector Comparison of Period July.9,2007-Oct.14,2008 and Oct. 15, 2008–Oct. 15, 2009, for multi-curve RFLMM
series of RMSPEs given the market prices and the model-implied prices of caplets/swaptions. There are two different pricing analysis methods. If the parameters are estimated from the same period as the term structure and the market prices, it is called in-sample pricing analysis, otherwise it is called out-of-sample analysis. The overall pricing performance results are summarized in Table 6.4 and 6.5. Table 6.4 and 6.5 show the root of mean square percentage error (RMSPE) over the in-sample and out-of-sample periods for the four different models, for caplets with all different maturities and swaptions with all maturities and lengths respectively. Now we first examine the pricing performance of the four different models and then investigate the relative valuation of caps and swaptions.

Table 6.4. RMSPEs of European Caps Valuation

<table>
<thead>
<tr>
<th></th>
<th>single-curve LMM</th>
<th>two-curve LMM</th>
<th>single-curve RFLMM</th>
<th>two-curve RFLMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>In-sample Average Errors</td>
<td>0.4245</td>
<td>0.3975</td>
<td>0.3051</td>
<td>0.2720</td>
</tr>
<tr>
<td>Out-of-sample Average Errors</td>
<td>0.5094</td>
<td>0.4770</td>
<td>0.3264</td>
<td>0.2937</td>
</tr>
</tbody>
</table>

Table 6.5. RMSPEs of European Swaptions Valuation

<table>
<thead>
<tr>
<th></th>
<th>single-curve LMM</th>
<th>two-curve LMM</th>
<th>single-curve RFLMM</th>
<th>two-curve RFLMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>In-sample Average Errors</td>
<td>2.3034</td>
<td>2.0324</td>
<td>1.1856</td>
<td>0.8468</td>
</tr>
<tr>
<td>Out-of-sample Average Errors</td>
<td>2.5594</td>
<td>2.2583</td>
<td>0.9409</td>
<td>1.3173</td>
</tr>
</tbody>
</table>

6.3.3.1 The pricing performance of different models. In this section, the RMSPE pricing errors of caps and swaptions are investigated for the four different models, for both in-sample and out-of-sample periods. The pricing errors (RMSPE), averaged of caplets/swaptions of all different maturities and lengths over the in-sample and out-of-sample periods are shown in Table 6.4 and 6.5 respectively, while the time series of pricing errors are described in Figure 6.4, 6.5, 6.6 and 6.7.

Table 6.4 and 6.5 show the average pricing errors (RMSPE) of caps/swaptions over the in-sample and out-of-sample periods for the four different models. One interesting observation we have is that even the single-curve RFLMM models have less
pricing errors than two-curve LMM models, both for in-sample and out-of-sample pricing. We see that for in-sample performance the two-curve models have smaller pricing errors than single-curve models, especially after the summer of 2008 when many financial companies bankrupted and collapsed. For out-of-sample performance the two-curve models have almost the same pricing errors than single-curve models at beginning, while the two-curve models over-performs the single-curve models as time goes on. Generally the two-curve models perform better than one-curve models as expected, both in-sample and out-of-sample pricing. It is nature that the two-curve models achieve less pricing errors than single-curve models, since two-curve models introduce more parameters, which can fit the data more accurately. The over-performance of two-curve models is proved in out-of-sample pricing. However, the difference is not very significant. The reason may be that the swaptions we consider here are based on swaps which are used as the main instruments to bootstrap the single curve. Random fields models have less pricing errors than the Brownian motion case, which coincides with what we found in calibration results. This is not surprising, since random fields models is the infinite-factor extension of the Brownian motion models, which will achieve higher pricing accuracy with more factors. Moreover, random fields modeling have other advantages, such as unnessessariness of prior determination of factor number and unnessessariness of re-calibration, as described in Sec.6.3.2.

We can also investigate the time series pricing errors of caps/swaptions over the in-sample and out-of-sample periods for the four different models. Figure 6.4 and 6.5 present the results for caps and Figure 6.6 and 6.7 present the results for swaptions. As illustrated, we can see that the RMSPEs are generally very small and there is no extreme outliers. The RMSPEs of random fields models are less than that of Brownian motion case during the entire periods, both in-sample and out-of-sample, while the two-curve models perform better than one-curve models insignificantly. For
the in-sample swaption valuation (Figure 6.6), the RMSPEs of two-curve models is more oscillated than that of one-curve models, while for cap valuation there is no significant difference. For in-sample cap valuation, the RMSPE errors take minimum values at around Apr.2008, while this is the time of maximum RMSPEs for in-sample swaption valuation (Figure 6.6). In fact this is an evidence of inconsistency of cap and swaption pricing. This issue will be discussed in details in next section. Another observation we have is that RFLMM models do not perform significant well than LMM model in out-of-sample case (Figure 6.5). This is also an evidence that swaption is a better choice of calibration instruments than cap, since cap price is more sensitive to the change of curves. From the out-of-sample performance of swaption valuation, we can see that the two-curve models over-perform the one-curve models after May. 2009. We can take this as an evidence that the out-of-sample performance of swaption is consistent that of in-sample performance.

![Figure 6.4. Time Series of RMSPEs of Caps for Single-curve LMM, RFLMM and Two-curve LMM, RFLMM Over The Period Jul.07-Oct.08(In-sample pricing)](image-url)
Figure 6.5. Time Series of RMSPE of Caps for Single-curve LMM, RFLMM and Two-curve LMM, RFLMM Over The Period Oct.08-Aug.09(Out-of-sample pricing)

Figure 6.6. Time Series of RMSPE of Swaptions for Single-Curve LMM, RFLMM and Two-curve LMM, RFLMM Over The Period Jul.07-Oct.08(In-sample pricing)
6.3.3.2 The relative valuation of swaptions and caps. In this section, the RMSPE pricing errors, as well as their statistics are studied for caps and swaptions to investigate the relative valuation of swaptions and caps. The pricing errors (RMSPE), averaged of caplets/swaptions of all different maturities and lengths over the in-sample and out-of-sample periods are shown in Table 6.4 and 6.5 respectively, while their statistics summary over two period (both in-sample and out-of-sample) are shown in Table 6.6 to 6.7 and Table 6.14 to 6.21, for ATM caps and ATM swaptions respectively, with mean as entries and standard errors in parentheses.

Table 6.4 and 6.5 show the average pricing errors (RMSPE) of swaptions over the in-sample and out-of-sample periods for the four different models. From these tables we can see that caps tend to have less pricing errors than swaptions. This is reasonable since the formulas of cap prices is much simpler than that of swaptions and there should be less error in the pricing and computing process. In addition, for both caps and swaptions, the in-sample pricing errors are smaller than the out-of-
sample pricing errors, while for caps, the in-sample pricing errors are greater than the out-of-sample pricing errors. The result of swaption pricing is reasonable since we use the in-sample data to estimate the parameters of the model. We state the reason that the caps has greater in-sample pricing errors as follows.

Beyond the overall RMSPEs, we can also examine the pricing errors for individual caps or swaptions. As pointed out in Longstaff et. al. [53], while the overall RMSPEs are small, the individual caps/swaptions may still exhibit high pricing errors. To study this possibility, we investigate the statistics summary over the whole period (both in-sample and out-of-sample) for ATM caps and ATM swaptions respectively. Table 6.6 to 6.7 present the results for caps and Table 6.14 to 6.21 present the results for swaptions. We see that the caplets with longer maturities have smaller RMSPE pricing errors, with smaller deviation and the swaptions with longer maturities have smaller RMSPE pricing errors, while the length has no significant influence on the pricing errors. This is consistent with the results of Zhao and Li [52].

In random fields models, once the correlation structure and the instantaneous volatility has been estimated from the data, the covariance of forward rates is fixed. As pointed out in Sec.2.2, the price of caps and swaptions will be determined once the RFLMM model is estimated. Since both the RFLMM closed-form formulas of cap and swaption contain the instantaneous volatility and the correlation structure, there should exist a relation between the cap prices and the swaption prices, given the same estimation results. Longstaff et. al. [53] investigated the relative valuation of caps and swaptions in string model, which is a discrete version of random fields model. They argued that the prices of caps at time $t$ in the model are implied from the prices of swaptions at time $t$ and they investigated the relative valuation of caps and swaptions empirically. In our random fields LMM model, which is the continuous version of Longstaff’s string model, we will also investigate the relation between cap
prices and swaption prices. However, in our setting we can not simply compare the market cap prices with model-implied prices, since our model is estimated from only term structure but not the swaption price. Given the fact that we have priced both the caps and swaptions using the same estimation results, we can check the serial correlation of model-implied cap prices and swaption prices. A high correlation may implied that the relation between cap prices and swaption prices is significantly strong. The correlation is only 0.25 for the in-sample period and 0.26 for the out-of-sample period, which means that the cap prices and swaption prices do not have significant relation between each other. This is consistent with the results of Longstaff et. al. [53] and other author’s conclusion, such as Hull and White [39].

Table 6.6. RMSPEs of Individual ATM Caplet Valuation (In-sample)

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<td></td>
<td>(0.2390)</td>
<td>(0.1662)</td>
<td>(0.1356)</td>
<td>(0.1153)</td>
<td>(0.0963)</td>
<td>(0.0881)</td>
<td>(0.0848)</td>
<td>(0.0848)</td>
<td>(0.0841)</td>
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<tr>
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<td>(0.1575)</td>
<td>(0.1516)</td>
<td>(0.1314)</td>
<td>(0.1132)</td>
<td>(0.1031)</td>
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<td>(0.0865)</td>
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<td>0.3025</td>
<td>0.2815</td>
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<tr>
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<td>(0.1400)</td>
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<td>0.2230</td>
<td>0.1868</td>
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<td>(0.1517)</td>
<td>(0.1440)</td>
<td>(0.1342)</td>
<td>(0.1278)</td>
<td>(0.1170)</td>
<td>(0.1158)</td>
<td>(0.1199)</td>
<td>(0.1201)</td>
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Table 6.7. RMSPEs of Individual ATM Caplet Valuation RMSPEs (Out-of-sample)

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<td>(0.1627)</td>
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<td>(0.1577)</td>
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<td>(0.1074)</td>
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<td>0.3139</td>
<td>0.2676</td>
<td>0.2242</td>
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<td>(0.1660)</td>
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<td>(0.1728)</td>
<td>(0.1610)</td>
<td>(0.1534)</td>
<td>(0.1404)</td>
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<td>(0.1439)</td>
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<tr>
<td>two-curve RFLMM</td>
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<td>0.3497</td>
<td>0.3227</td>
<td>0.2807</td>
<td>0.2408</td>
<td>0.2018</td>
<td>0.1634</td>
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<td>(0.1494)</td>
<td>(0.1638)</td>
<td>(0.1555)</td>
<td>(0.1449)</td>
<td>(0.1380)</td>
<td>(0.1264)</td>
<td>(0.1251)</td>
<td>(0.1295)</td>
<td>(0.1297)</td>
</tr>
</tbody>
</table>
6.3.4 Hedging performance. The idea of hedging is to create a portfolio of assets whose value goes in the opposite direction than the value of the holding position when market fluctuates. Thus to build a hedging strategy the derivative’s sensitivities to the underlying instruments need to be calculated. An option’s delta is defined as the rate of change of the option price with respect to the price of underlying instruments, which is calculated mathematically as the first derivatives of option’s value with respect to the underlying instruments. In this thesis we calculate the first derivative of at-the-money European cap/swaption price with respect to some underlying rates. From Eq.(1.22), we can define the delta of the caplet as follows,

\[ \Delta = \frac{\partial \text{Cplt}(L_k(t))}{\partial L_k(t)} = \frac{\text{Cplt}(L_k(t) + h) - \text{Cplt}(L_k(t) - h)}{2h}, \]  

(6.16)

where \( \text{Cplt}(t) \) is the price of caplet on \([T_{k-1}, T_k]\) and \( L_k(t) \) is the underlying forward rate. From Eq.(1.27), we can define the delta of the swaption as follows,

\[ \Delta = \frac{\partial \text{Swpt}(S_{i,j}(t))}{\partial S_{i,j}(t)} = \frac{\text{Swpt}(S_{i,j}(t) + h) - \text{Swpt}(S_{i,j}(t) - h)}{2h}, \]  

(6.17)

where \( \text{Swpt}(t) \) is the price of swaption with maturity \( T_i \) and length \( T_j - T_i \) and \( S_{i,j}(t) \) is the swap rate. Following Li and Zhao [52] and Wu [71], we consider the Hedging Variance Ratios (HVR) of at-the-money European caps and swaptions to compare the four LIBOR market models.

The HVR is defined as follows. At day \( t_1 \) we calculate the delta and at next day \( t_2 \) we use the delta got in day \( t_1 \) to hedge the derivatives. The hedging error after one day is thus defined as

\[ \text{HedgingError}_{t_1} = V(t_2) - \Delta_{t_1} F(t_2) - [V(t_1) - \Delta_{t_1} F(t_1)] \]

\[ = V(t_2) - V(t_1) - \Delta_{t_1} [F(t_2) - F(t_1)]. \]  

(6.18)

For example, if the swaption has maturity \( T_i \) and length \( T_j - T_i \) we use the swap rate \( S_{i,j}(t) \) as the underlying instrument. Since delta changes as time goes on, the hedge
strategy needs to be adjusted periodically. This is known as re-balancing. We re-
balance the delta daily and accumulate the hedging errors up to the hedging horizon. The accumulated hedging error after $m$ day is defined as

$$\text{AccuHedgingError} = \sum_{k=1}^{m} [V(t_{k+1}) - V(t_k) - \Delta t_k [F(t_{k+1}) - F(t_k)]]$$ \hspace{1cm} (6.19)

Given time series of swap rates and derivative prices, we will have the corresponding accumulated hedging errors. The HVR is defined as

$$HVR = 1 - \frac{\text{Var(AccuHedgingError)}}{\text{Var}(V(t))}.$$ \hspace{1cm} (6.20)

The higher the HVR, the better the hedging performance.

The time period chosen for hedging analysis is from July 16, 2007 to Aug. 1, 2009, which is the main period of credit crunch. We use the term structure and prices of at-the-money caplets/swaptions to compare the hedging performance of four models, single-curve LMM, single-curve RFLMM, two-curve LMM, two-curve RFLMM. The caps have maturities of 1,\ldots, 9 years and the swaptions have maturities of 0.5, 1, 2,\ldots, 7 years (except 6 years), lengths of 1, 2,\ldots, 9 years, with the sum of the maturity and swap length not exceeding ten years. In total, we have 9 cap prices and 47 swaption prices. We calculate HVR for each of caps/swaptions and thus for each model we have a table of HVRs. In addition, we calculate the mean of all HVRs for each model to better show the hedging performance of the four models. Table 6.8, 6.9, 6.10, 6.13 show the hedging results on period July 9, 2007 to Oct. 14, 2008.

Table 6.8 presents the hedging performance results (HVR) of at-the-money caps/swaptions, the average HVR of all caplets/swaptioins in the period. From the results, we can see that for both caps and swaptions, the two-curve RFLMM has the largest HVR and the single-curve LMM has the smallest HVR. In addition, the hedging performance of caps is better than that of swaptions. This result is reasonable, since the caps can directly use the forward rates as the underlying instrument
and thus easier to hedge. Table 6.9 shows the hedging performance (HVR) of ATM caps with different maturities for the four models. For hedging performance (HVR) of ATM swaptions with different maturities and lengths, we take single-curve LMM and two-curve RFLMM as examples, as shown in Table 6.10, 6.11, 6.12 and 6.13, respectively. We can see that the swaptions with shorter maturities and lengths tend to have higher HVR. The results of individual caps and swaptions performance coincide with the results of Li and Zhao [52] that long-term caps are hard to hedge.

Table 6.8. Hedging Performance (HVR) of ATM Caps and ATM Swaptions

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<th>Average HVR for Four Models(%)</th>
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<tr>
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<td>single-curve LMM</td>
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<tr>
<td>caps</td>
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<tr>
<td>swaptions</td>
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</table>

Table 6.9. Hedging Performance (HVR) of Individual ATM Caps

<table>
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<tbody>
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<td>0.9993</td>
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<td>0.9979</td>
<td>0.9937</td>
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</tr>
<tr>
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<td>0.9996</td>
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<td>0.9990</td>
<td>0.9987</td>
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<td>0.9996</td>
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Table 6.10. Hedging Performance (HVR) of Individual ATM Swaptions for single-curve LMM

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Table 6.11. Hedging Performance (HVR) of Individual ATM Swaptions for two-curve LMM

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Table 6.12. Hedging Performance (HVR) of Individual ATM Swaptions for single-curve RFLMM

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Table 6.13. Hedging Performance (HVR) of Individual ATM Swaptions for two-curve RFLMM

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Table 6.15. RMSPEs of Individual ATM Swaption Valuation for two-curve LMM (In-sample)

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Table 6.16. RMSPEs of Individual ATM Swaption Valuation for single-curve RFLMM (In-sample)

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Table 6.17. RMSPEs of Individual ATM Swaption Valuation for two-curve RFLMM (In-sample)

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CHAPTER 7
CONCLUSIONS

The goal of this thesis was to build a new LIBOR market model to explain the interest rates inconsistency after the 2008 credit crunch. The new model is an extension of LIBOR market model in multi-curve framework when interest rates are described as random fields.

In Chapter 1, we reviewed the LIBOR market model and provided the closed form formulas of Black implied volatilities for caplets and swaptions. In Chapter 2, we extended the LIBOR market model to random fields case (RFLMM), where the innovation terms were modeled as Gaussian fields. The derivatives prices were proved to satisfy a Black-Scholes type partial differential equation, which gives rise to closed form solutions for caplet and swaption prices. In Chapter 3, we derived the volatility smile models in random fields case. We took the lognormal mixture model as an example of local volatility models and took SABR type and Wu-Zhang type models as instances of stochastic volatility models. The approximated formulas of implied volatilities were thus obtained. In Chapter 4, all the previous work was extended to multi-curve framework, where the curve for discounting and the curve for future cash flow generating are modeled distinctly and jointly. The so called multi-curve random fields LIBOR market model (MRFLMM) was thus derived and the closed form formulas of Black implied volatilities for caplets and swaption were provided. The closed form implied volatilities formulas were used in calibration and estimation.

We consider the calibration and estimation in Chapter 5 and Chapter 6, respectively. This includes parameter estimation, pricing and hedging. We compare four different market models, single-curve LIBOR market model, single-curve RFLMM, two-curve LIBOR market model and two-curve RFLMM.
The calibration is based on spot market data on selected trading days before/during/after credit crunch. Both calibration results for cap surface and swaptions showed that the single-curve LIBOR market model and single-curve RFLMM fit the market data before credit crunch very well but did very badly for the data during/after credit crunch, while the two-curve LMM and RFLMM had more stable and accurate calibration results. On the other hand, generally the RFLMM (both single-curve or two-curve) had more stable and accurate results. For the calibration to cap surface, the random models had more accurate results but took more time, while for the calibration to swaptions, the RFLMM had more accurate results and took less time. The reason is that the calibration of RFLMM for caplets contains the correlation which may take more time and produce more accurate results than LIBOR market model, while the calibration of both LIBOR market model and RFLMM for swaptions contains the correlation and the form of correlation given by RFLMM has less parameters, which may save the computing time. In addition, the calibration results showed that the random fields volatility smile models had higher degree of capability of recovering general skews, especially when the volatility curve was more steep or showing non-zero slop for at-the-money level. The reason is that the calibration of random fields models contains more information from market, i.e., the correlation structure.

The estimation was performed using unscented Kalman filter on a time series of forward rates during credit crunch. It was shown empirically that random fields modeling has many advantages over Brownian motion modeling. It is not necessary to determine the number of factors prior to modeling and re-calibration is also not needed, since the covariance of forward rates in random field modeling is very stable within daily re-calibration and period estimation. We also consider the pricing and hedging performance. The results of in-sample and out-sample pricing showed that the accuracy of two-curve models was much better than that of single-curve models,
while accuracy of random field models is similar to that of standard setting. The reason may rely on the fact that the credit crunch causes the inconsistency of curves and the estimation of random fields model evolves the correlation matrix which may tie the price of caps to European swaptions more closely. The results of delta hedging showed that multi-curve models performed much better than single-curve models and the random fields models were not significantly better.
APPENDIX A

STOCHASTIC CALCULUS FOR RANDOM FIELDS
For fixed $T \in \mathbb{R}$, Brownian field $W(t,T)$ is a Brownian motion. Thus we can define stochastic calculus based on random fields for fixed $T$ in the same way as for Brownian motion. The Brownian motion analogy of the following definitions and theorems can be found in Oksendal [58] and Klebaner [51].

**Definition A.1.** [58] Let $(\Omega, \mathcal{F}, p)$ be a probability space. A function $Y : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}.$$  

See Oksendal [58] for the definition of probability space $(\Omega, \mathcal{F}, P)$.

**Definition A.2.** [58] Let $W(t,T)$ be a random field. Then we define $\mathcal{F}_t$ to be the $\sigma$-algebra generated by the random variables $\{W((s,\tau))\}_{0 \leq s \leq t, t \leq \tau \leq T}$. In other words, $\mathcal{F}_t$ is the smallest $\sigma$-algebra containing all sets of the form

$$\{\omega; W(t_1,T_1,\omega) \in F_{i1}, ..., W(t_j,T_j,\omega) \in F_{ij}\},$$

where $t_k \leq t$, $T_k \leq T$, $t_m \leq T_k$ for all $m, k$ and $F_{m,k} \subset \mathbb{R}$ are Borel sets, $k \leq j$, $m \leq i$.

**Definition A.3.** [58] Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an increasing family of $\sigma$-algebras of subsets of $\Omega$. For fixed $T \in \mathbb{R}$, a process $g(t,\omega) : [0,\infty) \to \mathbb{R}$ is called $\mathcal{F}_t$-adapted if for each $t \geq 0$, the function

$$\omega \to g(t,T,\omega)$$

is $\mathcal{F}_t$-measurable.

**Definition A.4.** [58] For every fixed $T \in \mathbb{R}$, let $\mathcal{V}(a,b)$ be the class of functions

$$\sigma(t,T,\omega) := \sigma(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

$(1)(t,\omega) \to \mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $[0,\infty)$.  

\( f(t, \omega) \) is \( \mathcal{F}_t \)-adapted.

(3) \( \mathbb{E}[\int_a^b \sigma(t, \omega)^2 dt] < \infty \).

**Definition A.5. Itō Integral with respect to Random Fields.** For every fixed \( T \in \mathbb{R} \), let \( \sigma \in \mathcal{V}(a,b) \). Then the Itō integral of \( \sigma \) for random fields is defined by

\[
\int_a^b \sigma(t, T, \omega)dW(t, T, \omega) = \lim_{n \to \infty} \int_a^b \phi_n(t, T, \omega)dW(t, T, \omega) \quad \text{(limit in } L^2(P))
\]

where \( \{\phi_n\} \) is a sequence of elementary functions such that

\[
\mathbb{E}\left[\int_a^b (\sigma(t, T, \omega) - \phi_n(t, T, \omega))^2 dt\right] \to 0, \quad \text{as } n \to \infty
\]

and \( \int_a^b \phi_n(t, T, \omega)dW(t, T, \omega) = \sum_{j \geq 0} e_j(\omega)[W(t_{j+1}, T) - W(t_j, T)](\omega) \) with \( \{t_1, ..., t_k, ...\} \) as a partition of \([a, b]\). Here \( e_j(\omega) \) has some further assumptions, see Oksendal [58].

**Theorem A.6. Martingale Representation Theorem for Random Fields.** Let \( W(t, T) \) be a random field. \( T \in [T_a, T_b] \). Suppose that \( M_t \) is an \( \mathcal{F}_t \)-martingale with respect to \( P \) and that \( M_t \in L^2(P) \) for all \( t \geq 0 \). Then there exists a unique stochastic process \( g(s, u, \omega) \) such that

\[
M_t = \mathbb{E}[M_0] + \int_{T_a}^{T_b} \int_0^t g(s, u, \omega)dW(s, u)du.
\]

**Definition A.7. Stochastic Differential Equation Driven by Random Fields.** [51] An equation of the form

\[
dX(t, T) = \mu(X(t), t, T)dt + \sigma(X(t), t, T)dW(t, T), \quad 0 \leq t \leq T, \quad X(0) = X_0, \quad (A.1)
\]

where functions \( \mu(x, t, T) \) and \( \sigma(x, t, T) \) are given and \( X(t, T) \) is the unknown process, is called a stochastic differential equation driven by random field \( W(t, T) \).

**Definition A.8. Strong Solutions to SDE’s Driven by Random Fields.**[51] Consider SDE of the form Eq.(A.1). A process \( X(t) \) is called a (strong) solution of SDE (A.1) if for \( t > 0 \), \( X(t) \) is a function of the given random field \( W(t, T) \), integrals \( \int_0^t \mu(X(s), s, T) \) and \( \int_0^t \sigma(X(s, T), s, T)dW(t, T) \) exist, and the following integral
equation is satisfied

\[ X(t, T) = X(0, T) + \int_0^t \mu(X(s, T), s, T)ds + \int_0^t \sigma(X(s, T), s, T)dW(s, T). \]  

(A.2)

In this thesis, when we refer to a solution of SDE, we mean strong solution.

**Theorem A.9. Existence and Uniqueness of Solutions for Stochastic Differential Equations Driven by Random Fields.**[58] For fixed \( T \in \mathbb{R} \) and \( \tau > 0 \), assume that \( \mu(\cdot, \cdot, \cdot) := \mu(\cdot, \cdot, T) : \mathbb{R} \times [0, \tau] \to \mathbb{R} \), \( \sigma(\cdot, \cdot, \cdot) := \sigma(\cdot, \cdot, T) : \mathbb{R} \times [0, \tau] \to \mathbb{R} \) are measurable functions satisfying

\[ |\mu(x, t)| + |\sigma(x, t)| \leq C(1 + |x|); x \in \mathbb{R}, t \in [0, \tau] \]  

(A.3)

for some constant \( C \), such that

\[ |\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq D|x - y|; x, y \in \mathbb{R}, t \in [0, \tau] \]  

(A.4)

for some constant \( D \). Let \( X_0 \) be a random variable which is independent of the \( \sigma \)-algebra generated by \( W(s, T, \cdot), s \geq 0 \) and such that

\[ \mathbb{E}[|X_0|^2] \leq \infty. \]

Then the SDE (A.1) has a unique \( t \)-continuous solution \( X(t, T, \omega) \) with the property that \( X(t, T, \omega) \) is adapted to the filtration \( \mathcal{F}_t \) generated by \( X_0 \) and \( (W(s, T, \cdot))_{s \leq t} \).

**Definition A.10. Quadratic Covariation.**[51] If \( X \) and \( Y \) are semimartingales on the common space, then the quadratic covariation process is defined by

\[ [X, Y](t) = \lim_{n \to \infty} \sum_{i=0}^{n-1} (X(t^n_{i+1}) - X(t^n_i))(Y(t^n_{i+1}) - Y(t^n_i)), \]  

(A.5)

where \( \{t^n_i\}_{i=0}^n \) is a partition of the interval \([0, t]\) and the limit is in probability when \( \delta_n = \max_i(t^n_{i+1} - t^n_i) \to 0 \).
APPENDIX B
PROOF OF THEOREMS AND COROLLARIES
Appendix B.1. Proof of Theorem 2.3.1:

Proof. Given the dynamics of the zero coupon bond price $P(t, T)$ in Eq.(2.13), by Itô’s formula, we have that

$$d\left(\frac{1}{P(t, T_k)}\right) = -\frac{r(t)dt - \int_t^{T_k} \sigma(t, u)d\tilde{W}(t, u)du + (\int_t^{T_k} \sigma(t, u)d\tilde{W}(t, u)du)^2}{P(t, T_k)},$$

and

$$dP(t, T_{k-1})d\left(\frac{1}{P(t, T_k)}\right) = -\frac{P(t, T_{k-1})d\left(\frac{1}{P(t, T_k)}\right) + dP(t, T_{k-1})d\left(\frac{1}{P(t, T_k)}\right)}{P(t, T_k)}$$

Thus the dynamics of $L_k(t)$ under risk neutral measure $Q$ can be derived as

$$dL_k(t) = d\left[\frac{1}{\delta_k}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right)\right]$$

$$= \frac{1}{\delta_k}\left[\frac{dP(t, T_{k-1})}{P(t, T_k)} + P(t, T_{k-1})d\left(\frac{1}{P(t, T_k)}\right) + dP(t, T_{k-1})d\left(\frac{1}{P(t, T_k)}\right)\right]$$

$$= \frac{1}{\delta_k} P(t, T_{k-1}) \left[\int_{T_{k-1}}^{T_k} \sigma(t, u)d\tilde{W}(t, u)du + \int_t^{T_k} \sigma(t, u)d\tilde{W}(t, u)du \right.$$

$$\left. + \int_{T_{k-1}}^{T_k} \sigma(t, u)d\tilde{W}(t, u)du\right].$$

Now we derive the dynamics of forward rates $L_k(t)$ under $T_k$-forward measure. Suppose that there exists a function $\theta(t, T_k, u)$ such that $dW^{T_k}(t, u) := \theta(t, T_k, u)dt + d\tilde{W}(t, u)$ has normal distribution $\Phi(0, dt)$ under $T_k$-forward measure. Replace $d\tilde{W}(t, u)$ by $dW^{T_k}(t, u) - \theta(t, T_k, u)dt$ in above formula, we obtain

$$dL_k(t) = \frac{1}{\delta_k} P(t, T_{k-1}) \left\{\int_{T_{k-1}}^{T_k} \sigma(t, u)[dW^{T_k}(t, u) - \theta(t, T_k, u)dt]du + \int_t^{T_k} \sigma(t, u)\right.$$

$$\left.[dW^{T_k}(t, u) - \theta(t, T_k, u)dt]du \int_{T_{k-1}}^{T_k} \sigma(t, u)[dW^{T_k}(t, u) - \theta(t, T_k, u)dt]du\right\}.$$

Since $L_k(t)$ is a martingale under $T_k$-forward measure, the drift term should vanish, i.e.

$$\int_{T_{k-1}}^{T_k} \sigma(t, u)\theta(t, T_k, u)dt du = \int_{T_{k-1}}^{T_k} \sigma(t, u)dW^{T_k}(t, u)du \int_{T_{k-1}}^{T_k} \sigma(t, v)dW^{T_k}(t, v)dv$$

$$= \int_{T_{k-1}}^{T_k} \sigma(t, u)[\int_t^{T_k} \sigma(t, v)c(u, v)dt dv] du,$$
which means that
\[ \theta(t, T_k, u) = \int_t^{T_k} \sigma(t, v)c(u, v)dv. \]

This completes the proof. \qed

\textbf{Appendix B.2. Proof of Theorem 2.3.4:}

\textit{Proof.} From Eq. (2.16) and Corollary 2.3.2, we know that
\[ dW_{T_k}(t, u) - \int_t^{T_k} \sigma(t, v)c(u, v)dvdt = dW(t, u) = dW_{T_{k+1}}(t, u) - \int_t^{T_{k+1}} \sigma(t, v)c(u, v)dvdt. \]

Thus
\[ dW_{T_k}(t, u) = dW_{T_{k+1}}(t, u) - \int_{T_k}^{T_{k+1}} \sigma(t, v)c(u, v)dvdt, \]
from which follows that for \( j > k \)
\[ dW_{T_k}(t, u) = dW_{T_j}(t, u) - \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \sigma(t, v)c(u, v)dvdt. \]  

(B.1)

Plugging Eq. (B.1) into Eq. (2.16), we obtain
\[ dL_k(t) = \frac{1}{\delta_k} (\delta_k L_k + 1)[\int_{T_{k-1}}^{T_k} \sigma(t, u)dW^T_j(t, u)du \]
\[ - \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \sigma(t, u) \int_{T_{i-1}}^{T_i} \delta_k \sigma(t, v)c(u, v)dvdu dt]. \]

As we know, \( L_k(t) \) is a martingale under \( T_k \)-forward measure. By random field martingale representation theorem, there exists a function \( \xi(t, u) \) such that
\[ dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)dW^T_k(t, u)du. \]  

(B.2)

From Eq. (2.16), we can simply take
\[ \xi_k(t, u) = \frac{\delta L_k(t) + 1}{\delta L_k(t)} \sigma(t, u). \]  

(B.3)
Thus Eq.(B.2) becomes

\[ dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)dW^j(t, u)du \]

\[ - \sum_{i=k+1}^{j} \frac{L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) \left( \frac{\delta L_i(t) \xi_i(t, v)c(u, v)}{\delta L_i(t)} + 1 \right) dvdt}{\delta L_i(t)}du \]

\[ = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^j(t, u) + \sum_{i=k+1}^{j} \frac{\delta L_i(t) \xi_i(t, v)c(u, v)}{\delta L_i(t)} + 1]dvdt]du. \]

The derivation in case \( j < k \) is perfectly analogous. The existence and uniqueness of solution \( L_k(t) \) is assured by the existence and uniqueness of \( f(t, T) \) in Eq.(1.11). By Theorem A.9, given the coefficients satisfying the required conditions, locally bounded, locally Lipschitz continuous and predictable, there exists a unique \( f(t, T) \) for Eq.(1.11). This completes the proof.

\[ \square \]

**Appendix B.3. Proof of Theorem 2.3.5:**

**Proof.** If the value of \( dW^T(t, u) \) on \([T_{k-1}, T_k] \) is \( dW^j_k(t) \), then

\[ \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_i} c(u, v)dvdu = \delta_k \delta_i c(t, T_k, T_i), \]

which means that for \( j < k \), Theorem 2.3.4 becomes

\[ dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u)[dW^j(t, u) + \sum_{i=j+1}^{k} \frac{\delta L_i(t) \xi_i(t, v)c(u, v)}{\delta L_i(t)} + 1]dvdt]du \]

\[ = L_k(t) \delta_i \xi_k(t, u)[dW^j(t, u) + \sum_{i=j+1}^{k} \frac{\delta_i L_i(t) \delta_i \xi_i(t, v)c(T_i, T_k)}{\delta_i L_i(t)}]dt]. \]

From Eq.(1.18), we can easily take \( c(t, T_i, T_k) = \rho_{i,k} \) and take \( \delta_k \xi_k(t) \) to be \( \xi_k(t) \). The derivation for \( j > k \) is analogous. Thus Eq.(2.18) reduces to Eq.(1.17) and Theorem 2.3.4 reduces to Eq.(1.18), which means that Eq.(1.18) is a discrete case of Theorem 2.3.4. This completes the proof. \[ \square \]
APPENDIX C

PROOF OF RANDOM FIELDS BLACK-SCHOLES EQUATION
Proof of Theorem 2.4.1:

Proof. We provide the proof of Black-Scholes equation with random fields for time dependent parameters. Suppose that we have an option $V$ on some underlying asset $S$, which has dynamics

$$
\frac{dS(t)}{S(t)} = \mu(t)dt + \int_{t_1}^{t_2} \sigma(t,u)dW(t,u)du. \tag{C.1}
$$

We create a portfolio $\Pi$ which consists of long $\Delta$ number of the asset and short one options, i.e.

$$
\Pi = -V + \Delta S.
$$

The increment of portfolio value is given by that of option and underlying asset, i.e.

$$
d\Pi = -dV + \Delta dS.
$$

By Itô’s formula for $dV$ and the dynamics of asset $S$, we can have that

$$
d\Pi = -dV + \Delta dS,
$$

where

$$
\hat{\sigma}^2(t,t_1,t_2) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sigma(t,u)\sigma(t,v)c(u,v)dudv. \tag{C.2}
$$

The portfolio will become non-stochastic if we choose $\Delta = \frac{\partial V}{\partial S}$, which means that the portfolio will grow in risk free interest rate $r(t)$:

$$
d\Pi = (-\frac{\partial V}{\partial t} - \frac{1}{2} \hat{\sigma}^2(t,t_1,t_2)S^2 \frac{\partial^2 V}{\partial S^2})dt = r(t)(-V + S\frac{\partial V}{\partial S})dt.
$$

By rearranging the equation above we can have the Black-Scholes equation with random fields for time dependent parameters:

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \hat{\sigma}^2(t,t_1,t_2)S^2 \frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0.
$$
We can introduce new parameters $Y = \ln S$, $\tau = T - t$, $U = e^{\int_t^\tau r(u)du}V$ and the above equation becomes

$$\frac{\partial U}{\partial \tau} = \frac{1}{2} \int_{t_1}^{t_1} \sigma(t, u) \sigma(t, v) c(u, v) dudv \frac{\partial^2 U}{\partial Y^2} + [r(\tau) - \int_{t_1}^{t_1} \sigma(t, u) \sigma(t, v) c(u, v) dudv] \frac{\partial U}{\partial Y},$$

which has fundamental solution as

$$\Phi(Y, \tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \hat{\sigma}^2(u, t_1, t_2)du}} e^{-\frac{1}{2} [Y + \int_0^\tau (r(u) - \hat{\sigma}^2(u, t_1, t_2)/2)du] / [2 \int_0^\tau \hat{\sigma}^2(u, t_1, t_2)du]}.$$

Thus the solution of Black-Scholes equation is given by

$$U(Y, \tau) = \int_{-\infty}^{+\infty} U(v, 0) \Phi(Y - v, \tau) dv.$$

The terminal conditions $V(S, T) = \max(S_T - K, 0)$ for call option give the price for a call option as

$$C = SN(\tilde{d}_1) - Ke^{-\int_0^\tau r(u)du}N(\tilde{d}_2),$$

where

$$\tilde{d}_1 = \frac{1}{\sqrt{\int_0^\tau \hat{\sigma}^2(u, t_1, t_2)du}} \left[ \ln \frac{S}{K} + \int_0^\tau (r(u) + \frac{\hat{\sigma}^2(u, t_1, t_2)}{2})du \right], \quad (C.3)$$

$$\tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_0^\tau \hat{\sigma}^2(u, t_1, t_2)du}. \quad (C.4)$$

This completes the proof.
APPENDIX D
STOCHASTIC TAYLOR EXPANSION METHOD TO SOLVE STOCHASTIC
DIFFERENTIAL EQUATIONS
The following theory is given in details in Kloeden et al. [48].

Consider a scalar Itô stochastic differential equation

\[ dX_t = f(t, X_t)dt + g(t, X_t)dW(t), \quad (D.1) \]

where \( W(t) \) is a standard scalar Wiener process. Define differential operator \( L, K \) by

\[ L = \frac{\partial}{\partial t} + f \frac{\partial}{\partial x} \frac{1}{2} g^2 \frac{\partial^2}{\partial x^2}; K = g \frac{\partial}{\partial x}. \]

By Itô's formula, for scalar-valued functions \( f(s, X(s)) \) and \( g(s, X(s)) \), we have that

\[ df(s, X(s)) = Lfds + Kgds + KgW(s). \quad (D.2) \]

The Euler-Maruyama scheme is derived by expanding coefficients using Itô's formula:

\[ X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + g(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW(s) \]

with remainder

\[ R_1(t_{n+1}, t_n) = \int_{t_n}^{t_{n+1}} Lf + \int_{t_n}^{t_{n+1}} Kgds + \int_{t_n}^{t_{n+1}} [\int_{t_n}^{t_{n+1}} Lg + \int_{t_n}^{t_{n+1}} KgW(u)]dW(s). \]

The next Taylor scheme, the Milstein scheme is derived by expanding \( Kg \) in \( R_1 \) by Itô's formula:

\[ X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + g(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW(s) \]

\[ + Kg(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} dW(u)dW(s) \]

\[ = X_{t_n} + f(t_n, X_{t_n}) \Delta t_n + g(t_n, X_{t_n}) \Delta W(t_n) + Kg(t_n, X_{t_n}) \frac{1}{2} (\Delta W(t_n)^2 - \Delta t_n), \quad (D.3) \]

with remainder \( R_2(t_{n+1}, t_n) \). General stochastic Taylor schemes can be formulated by applying Itô's lemma to remainders.
Now we would like to extend the stochastic Taylor expansion method to random field case. Consider
\[ dX_t = f(t, X_t)dt + \int_{t_1}^{t_2} g(t, X_t, u)dW(t, u)du := f(t, X_t)dt + g(t, X_t) \circ dW(t), \tag{D.4} \]
where \( W(t, u) \) is a Gaussian random field. Define differential operator \( L, K \) as
\[ L = \frac{\partial}{\partial t} + f \frac{\partial}{\partial x} + \frac{1}{2} \int_{t_1}^{t_2} g(u)g(v)dudv \frac{\partial^2}{\partial x^2}; K = g \frac{\partial}{\partial x}. \]

We can approximate the integral by
\[ \sum_{t_1 \leq u \leq t_2} \delta_n g(t, X_t, u)dW(t, u), \]
which gives rise to the stochastic Taylor expansion method:

The Euler-Maruyama scheme:

\[
X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + \sum_{t_1 \leq u \leq t_2} \delta_n g(t_n, X_{t_n}, u) \int_{t_n}^{t_{n+1}} dW(s, u)
\]

The Milstein scheme:

\[
X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + \int_{t_1}^{t_2} g(t, X_{t_n}, \bar{\sigma}) \int_{t_n}^{t_{n+1}} dW(s, \bar{\sigma})d\bar{\sigma} \\
+ \int_{t_1}^{t_2} \int_{t_1}^{t_2} g(t, X_{t_n}, \bar{\sigma}) \int_{t_n}^{t_{n+1}} dW(u, \bar{u})d\bar{u} \int_{t_n}^{t_{n+1}} dW(s, \bar{\sigma})d\bar{\sigma}
\]

\[
= X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + \sum_{t_1 \leq u \leq t_2} \delta_n g(t_n, X_{t_n}, u) \int_{t_n}^{t_{n+1}} dW(s, u)
\]

\[
+ \sum_{t_1 \leq u \leq t_2} \sum_{t_1 \leq u \leq t_2} \delta_n Kg(t_n, X_{t_n}, \bar{\pi}) \int_{t_n}^{t_{n+1}} dW(u, \bar{u}) \int_{t_n}^{t_{n+1}} dW(s, \bar{\pi})
\]

\[
= X_{t_n} + f(t_n, X_{t_n}) \Delta t_n + \sum_{t_1 \leq u \leq t_2} \delta_n g(t_n, X_{t_n}, u) \Delta W(t_n, u)
\]

\[
+ \sum_{t_1 \leq u \leq t_2} \delta_n Kg(t_n, X_{t_n}, u)[\sum_{t_1 \leq s \leq t_2} \delta_n \frac{1}{2}(\Delta W(t_n, s)^2 - \Delta t_n)]. \tag{D.5}
\]

General stochastic Taylor schemes can be formulated by applying Itô’s formula to remainders.
BIBLIOGRAPHY


